Markov Decision Processes

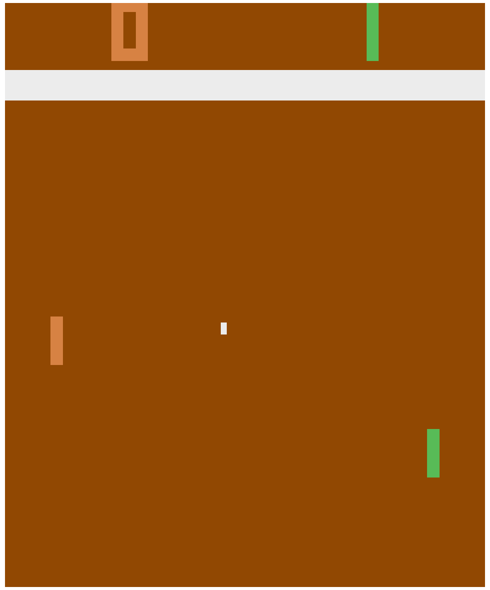
CMPT 729 G100

Jason Peng

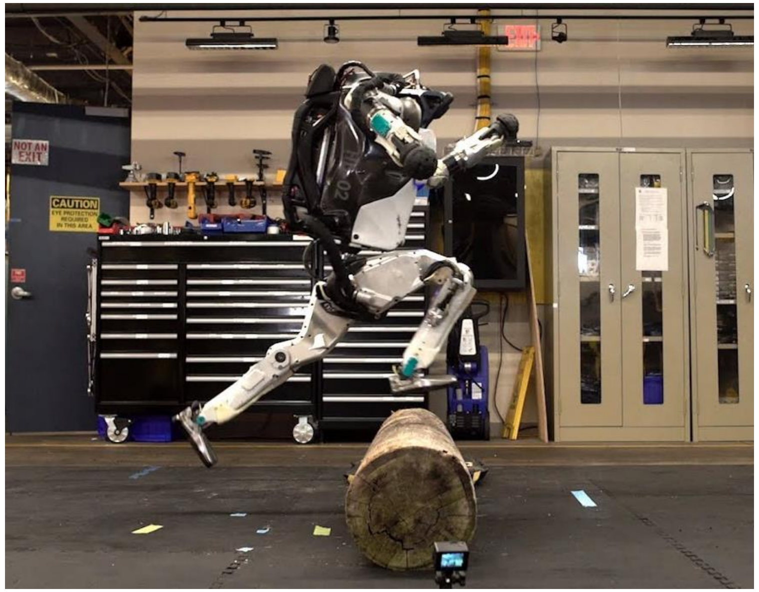
Overview

- Agent-Environment Interface
- Markov Decision Processes
- Partially Observable Markov Decision Processes

Environment Interaction



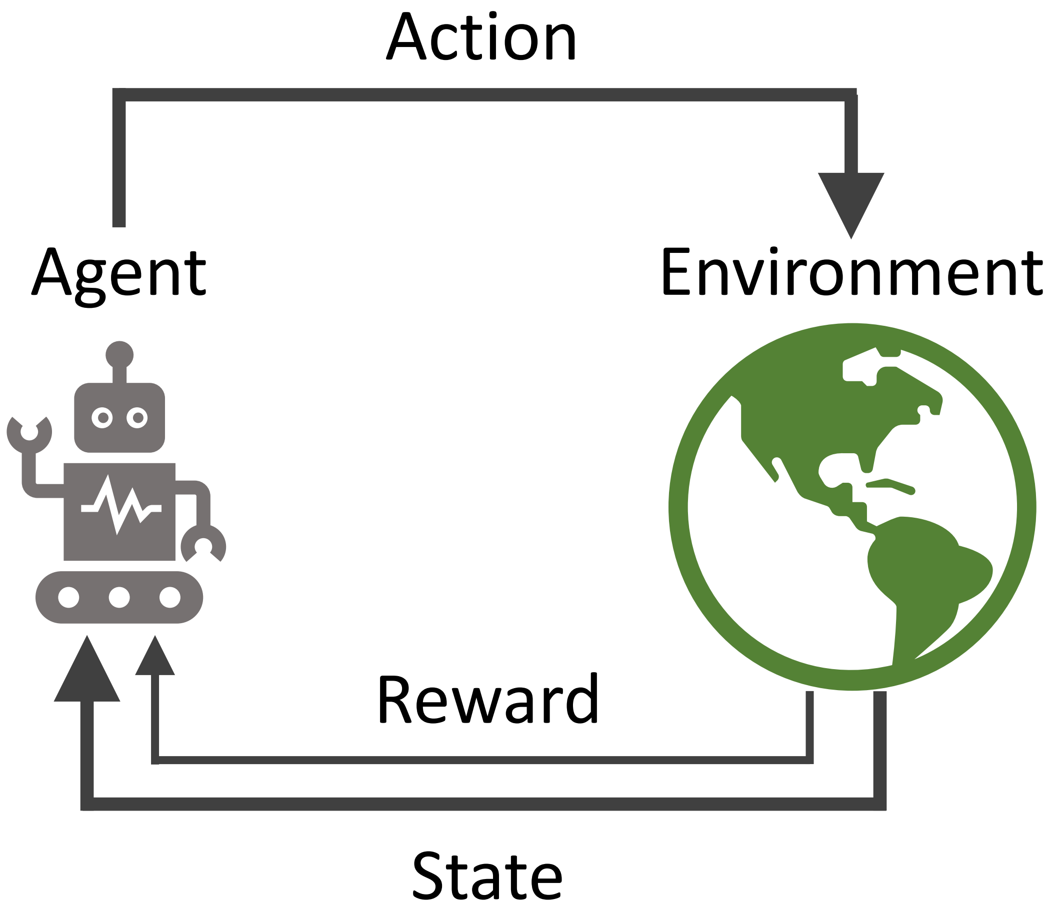
Pong [Atari]



Atlas [Boston Dynamics]

[Smithsonian Magazine]

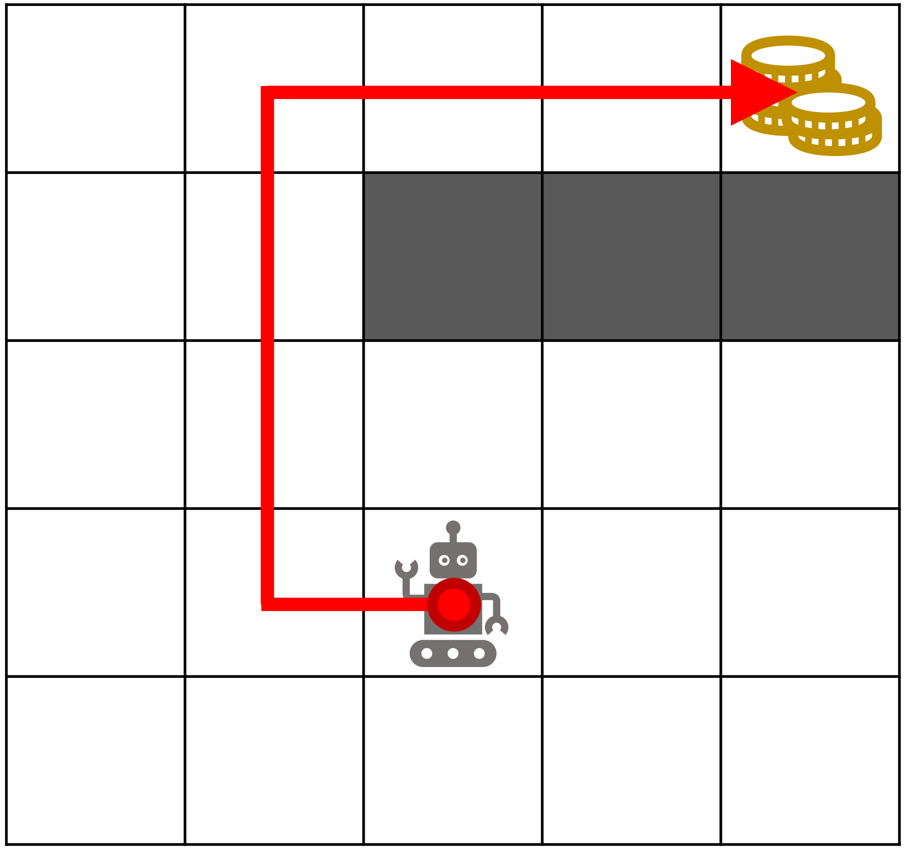
Agent-Environment Interface



Maze

State:

- position



Maze

State:

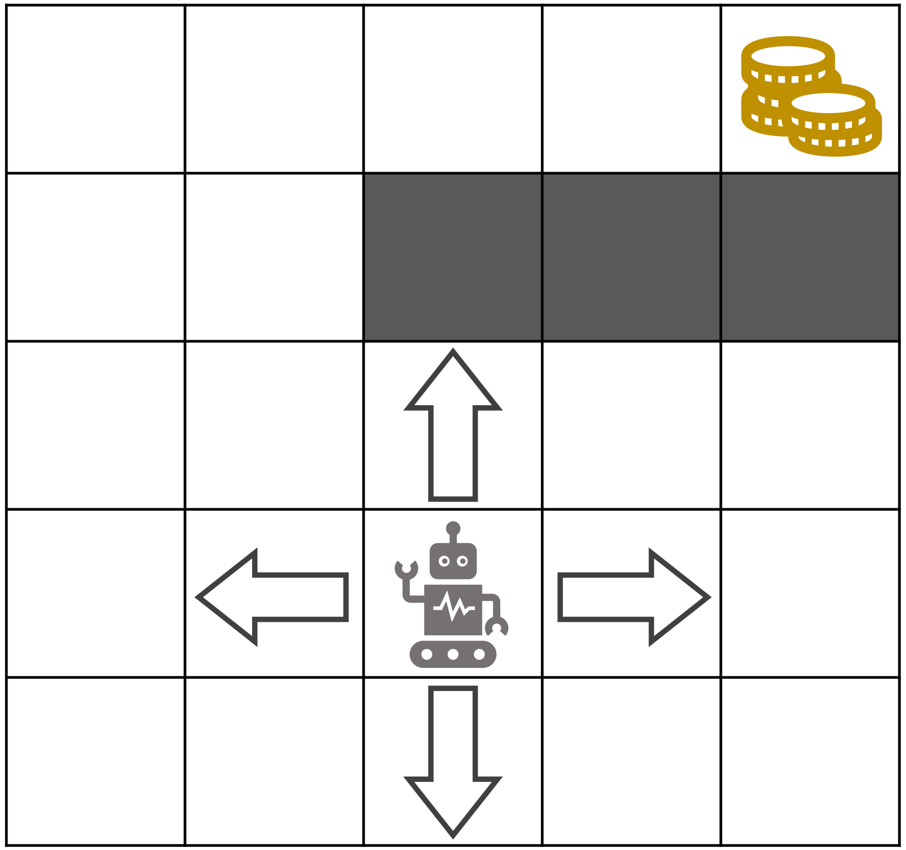
- position

Action:

- up / down / left / right / stop

Reward:

- 1 if goal reached
- 0 otherwise

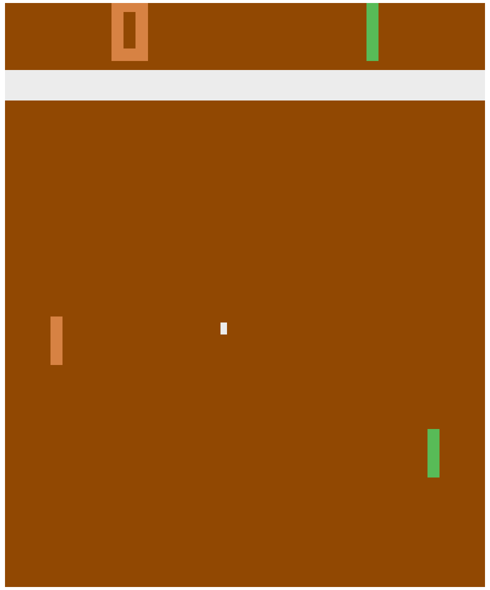


Pong

State:
- position + velocity of paddles
- position + velocity of the ball
- scores

Action:
- up / down / stop

Reward:
- 1 if agent scores
- -1 if opponent scores



Pong [Atari]

Humanoid Walking

State:

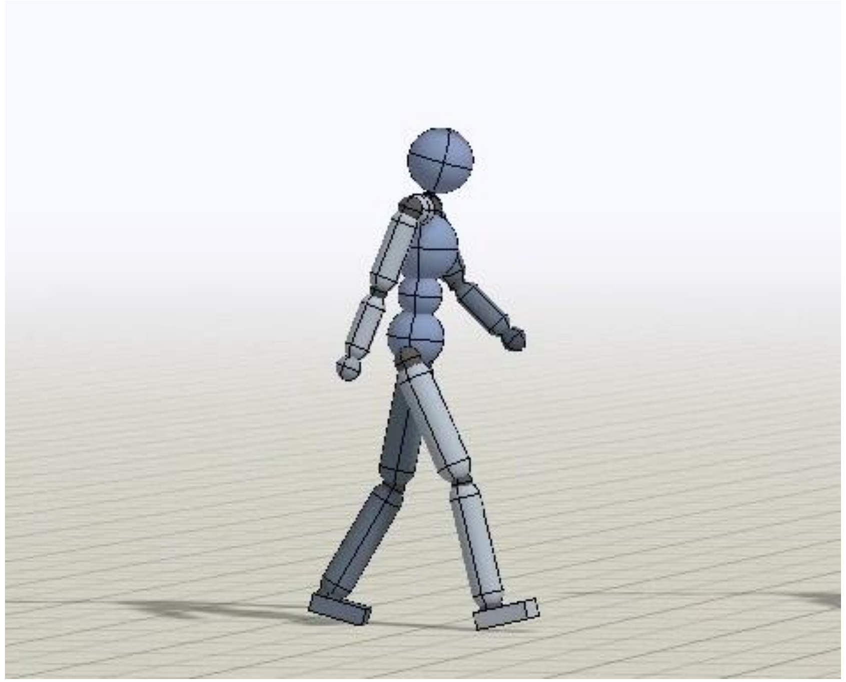
- position + velocity of body parts

Action:

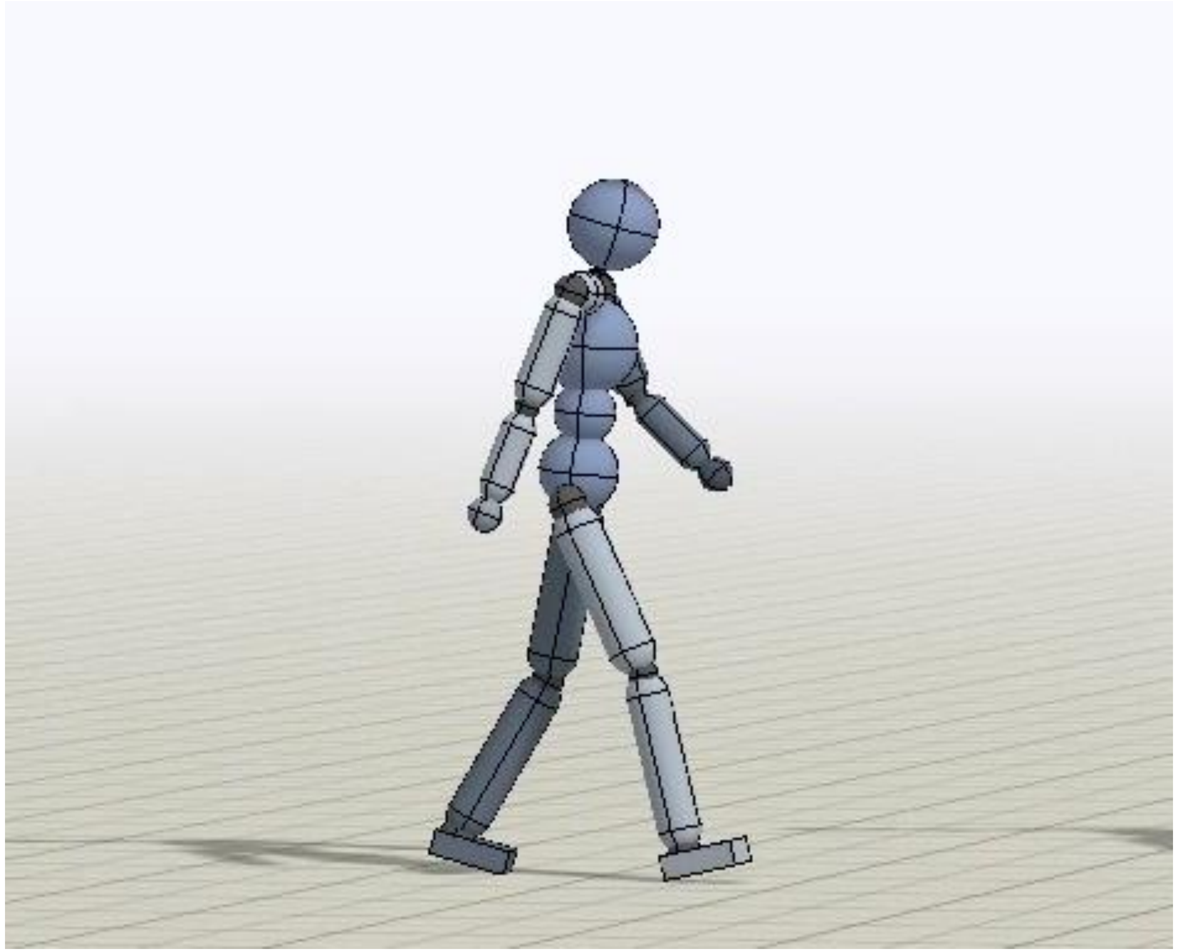
- motor forces

Reward:

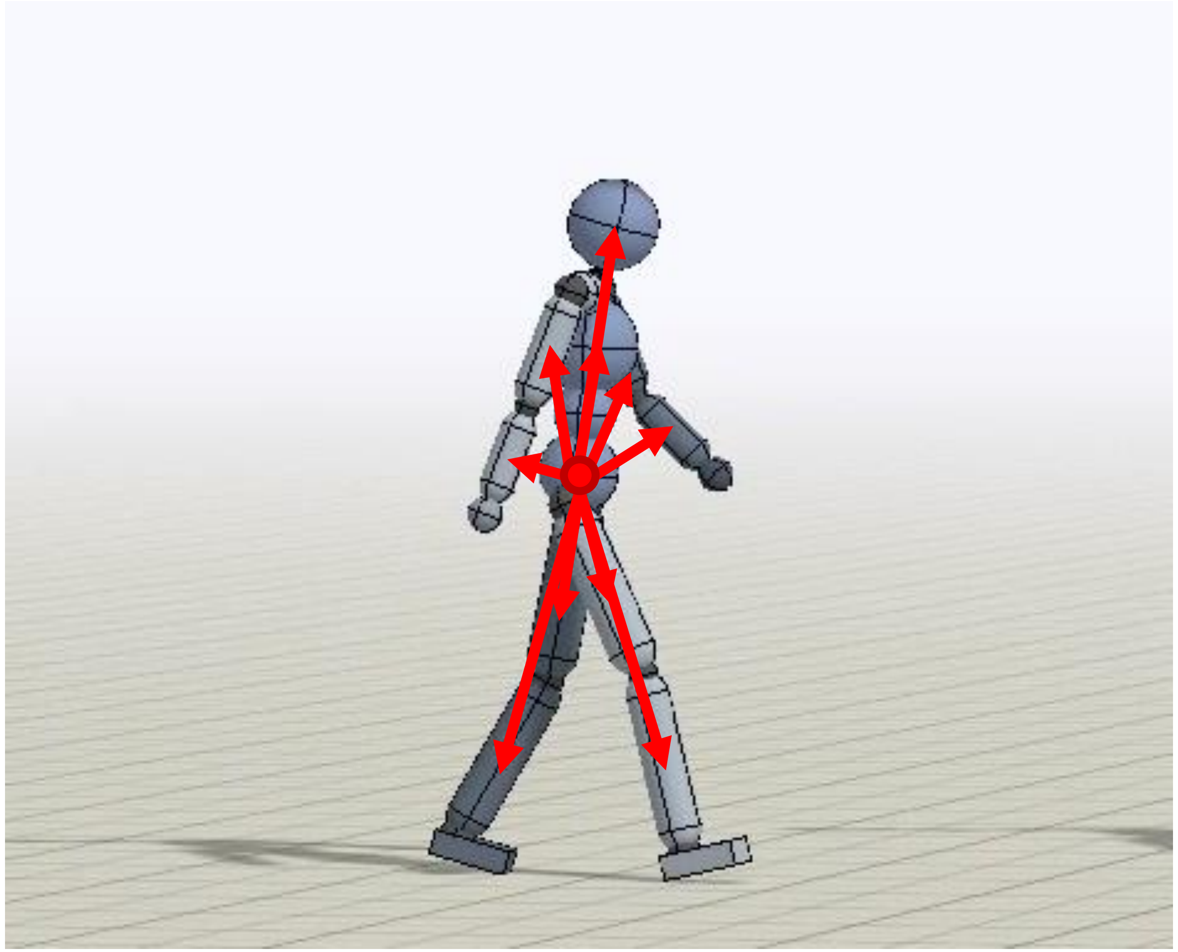
- speed



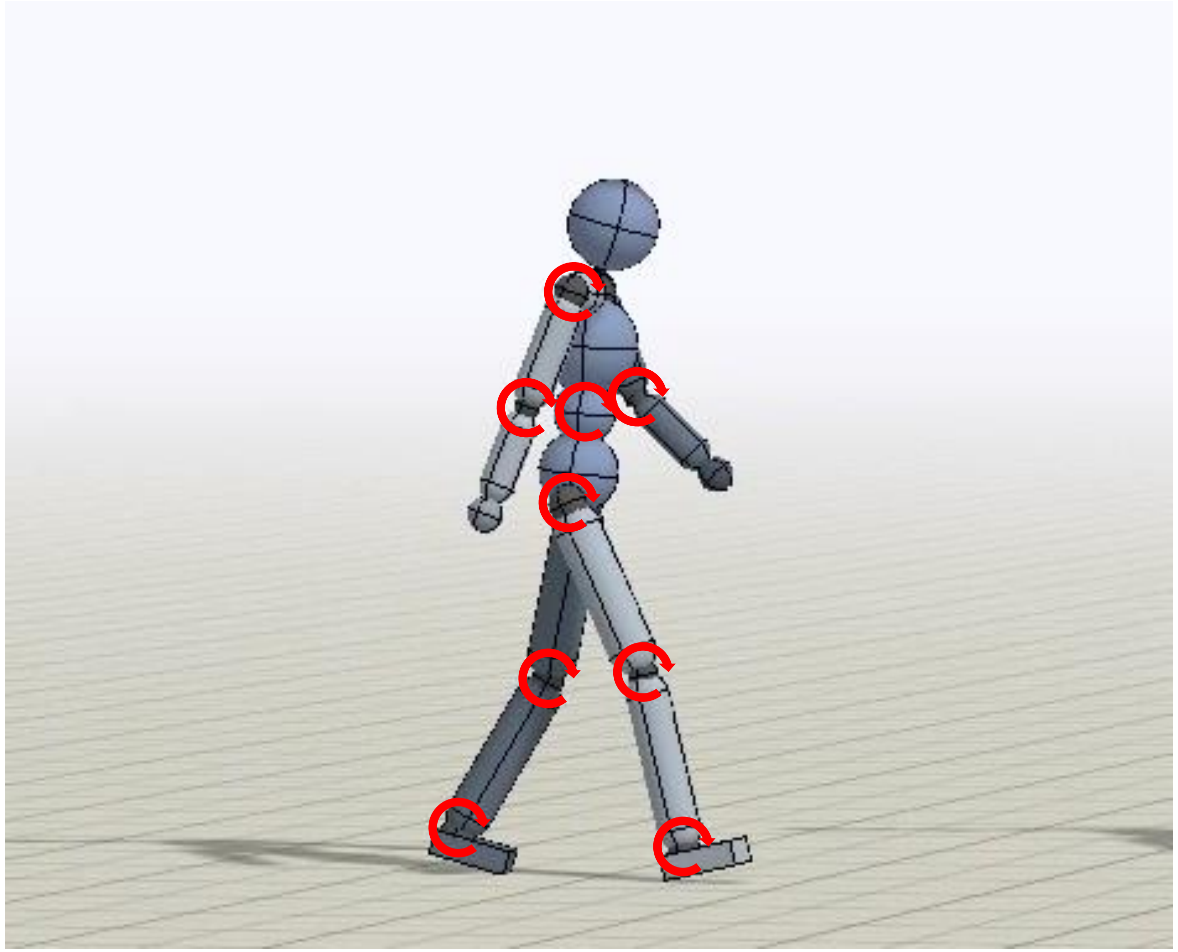
Humanoid Walking



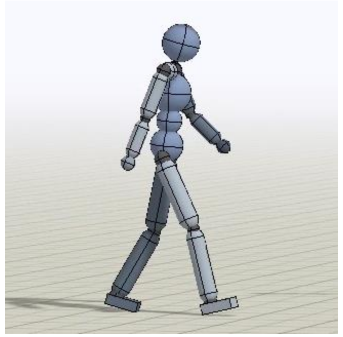
Humanoid Walking (State)



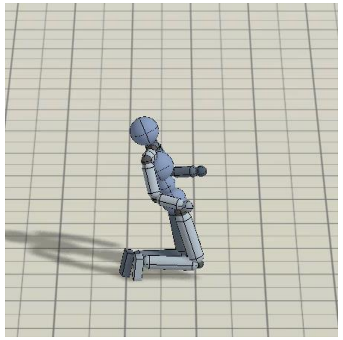
Humanoid Walking (Action)



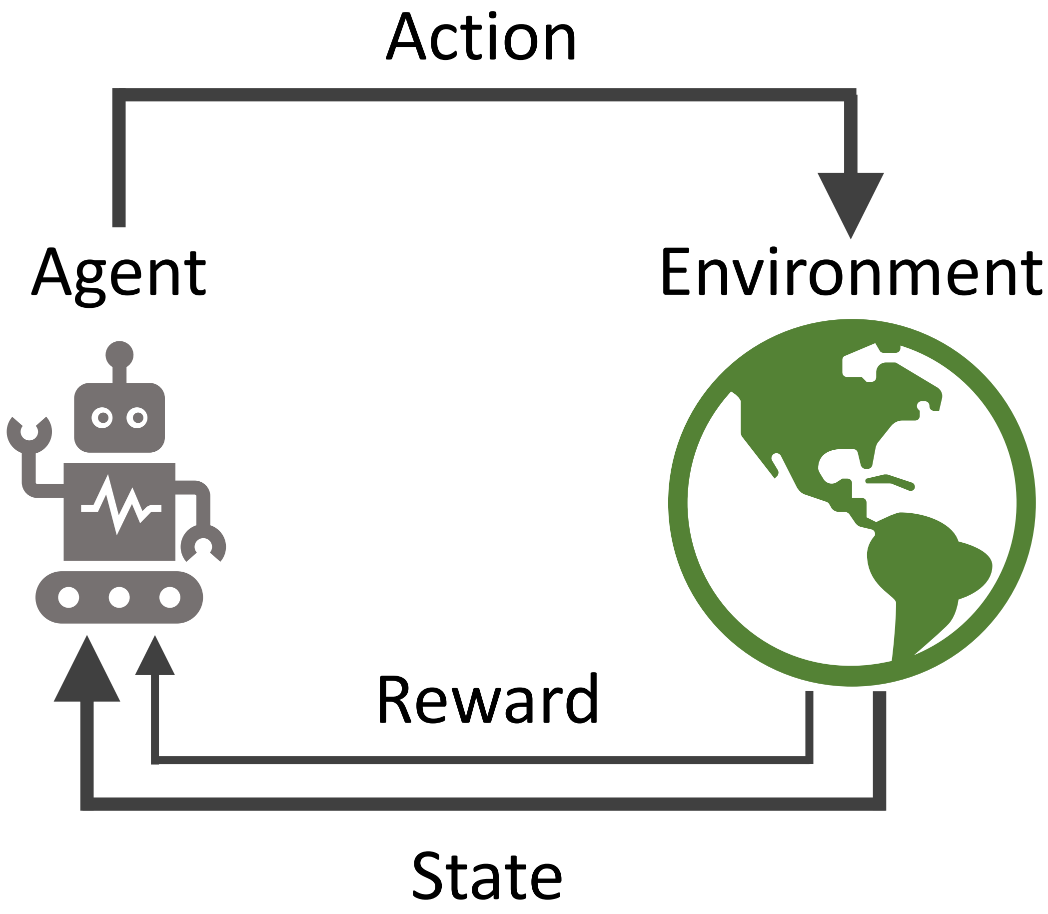
Humanoid Walking (Reward)

reward () = 

Humanoid Walking (Reward)

reward () = 

Agent-Environment Interface



Policy

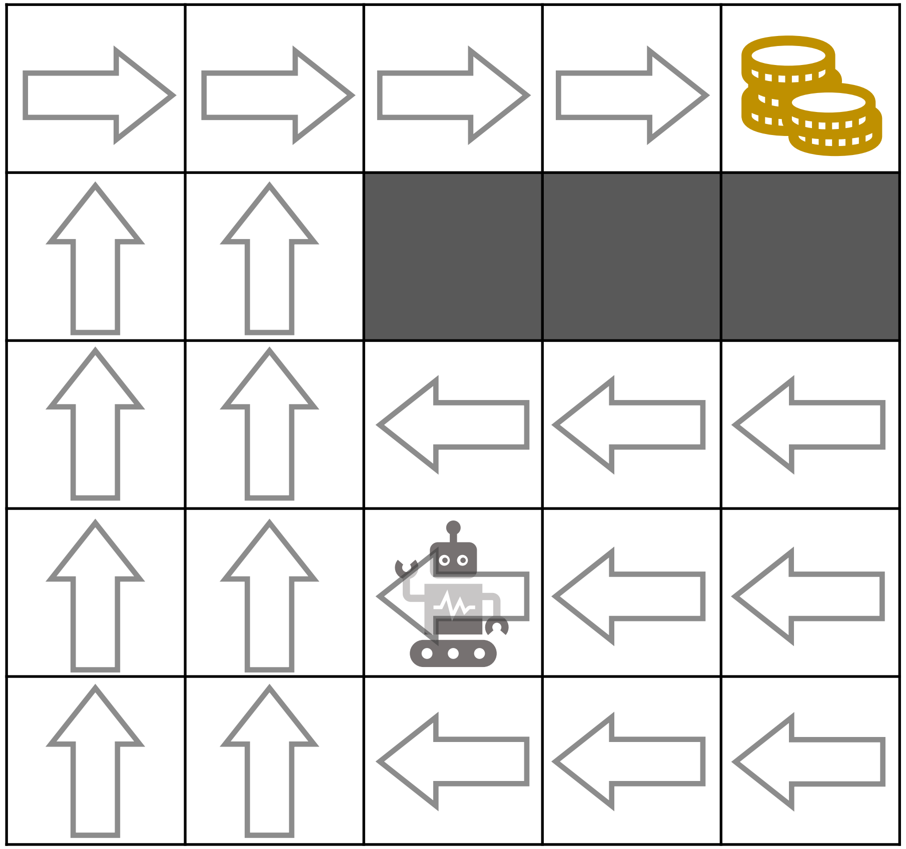
$$\mathbf{a} = \underline{\pi}(\mathbf{s})$$

$$\mathbf{s} \Rightarrow \boxed{\pi} \Rightarrow \mathbf{a}$$

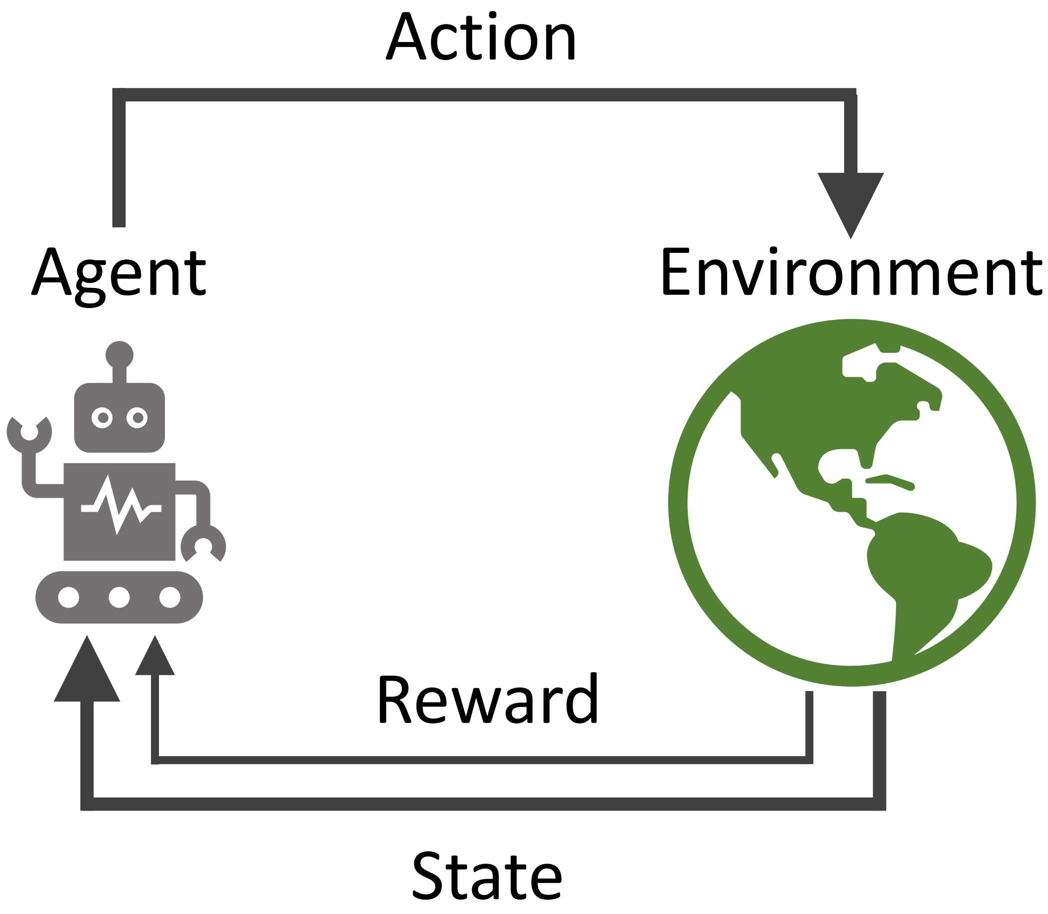
Policy

$$\mathbf{a} = \underline{\pi}(\mathbf{s})$$

Maze



Agent-Environment Interface



Markov Decision Process

$\mathbf{s} \in \mathcal{S}$ – state space

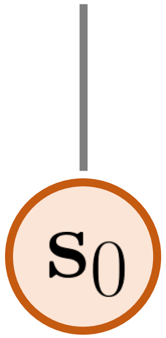
$\mathbf{a} \in \mathcal{A}$ – action space

$p(\mathbf{s}'|\mathbf{s}, \mathbf{a})$ – dynamics function

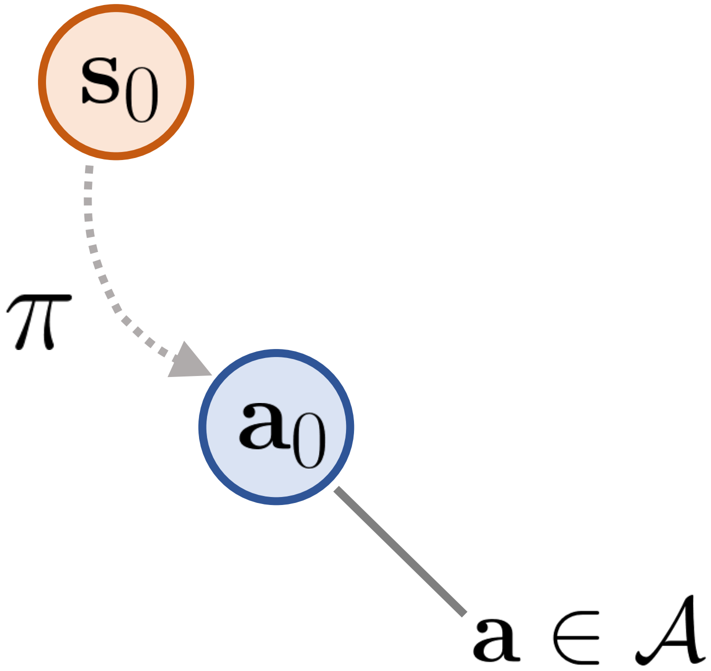
$r(\mathbf{s}, \mathbf{a}, \mathbf{s}')$ – reward function

MDP

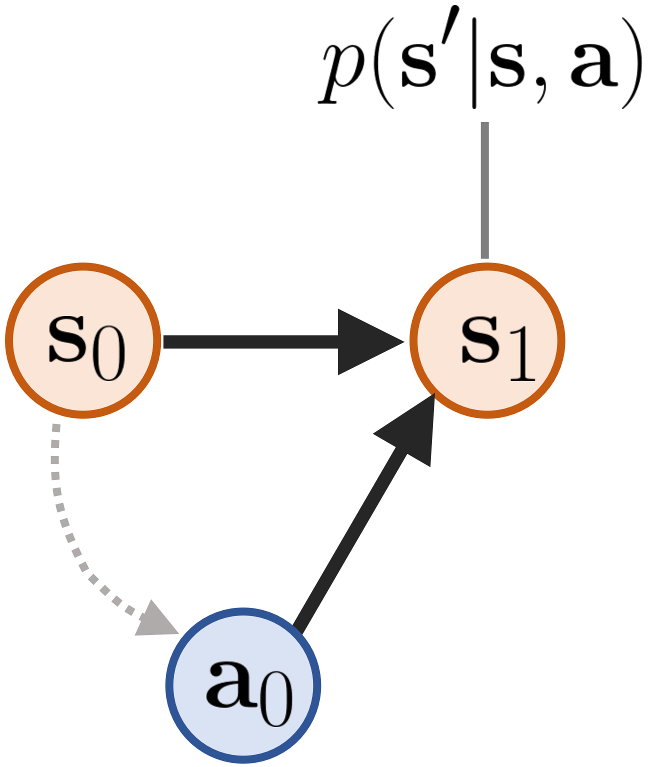
$\mathbf{s} \in \mathcal{S}$



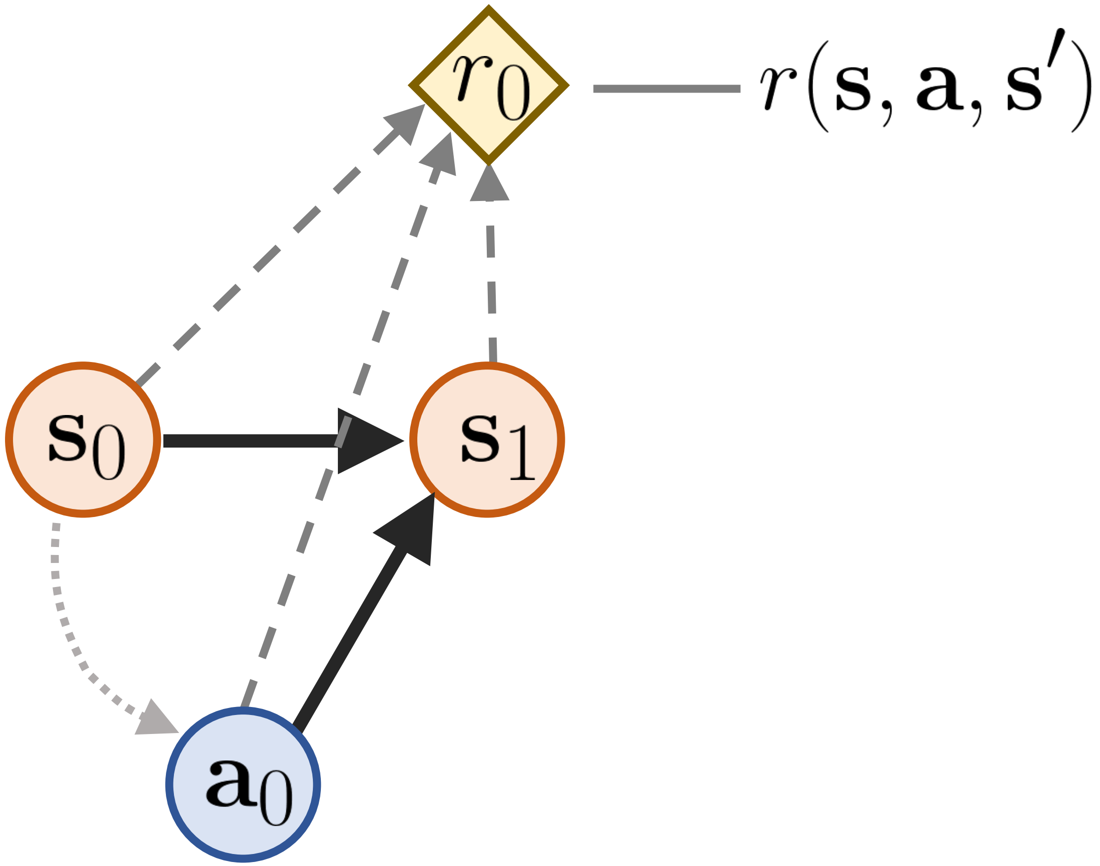
MDP



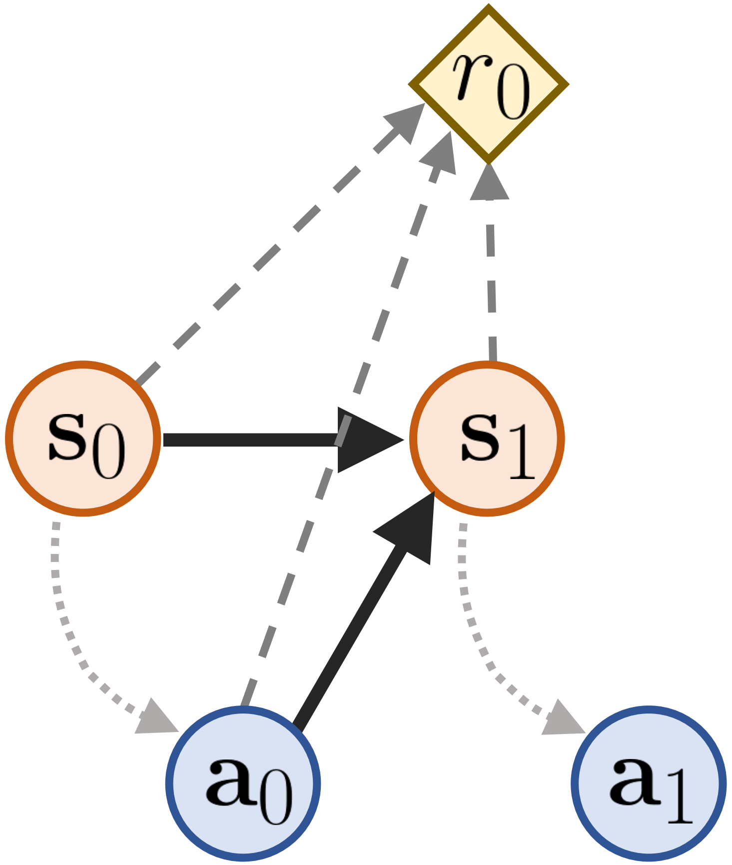
MDP



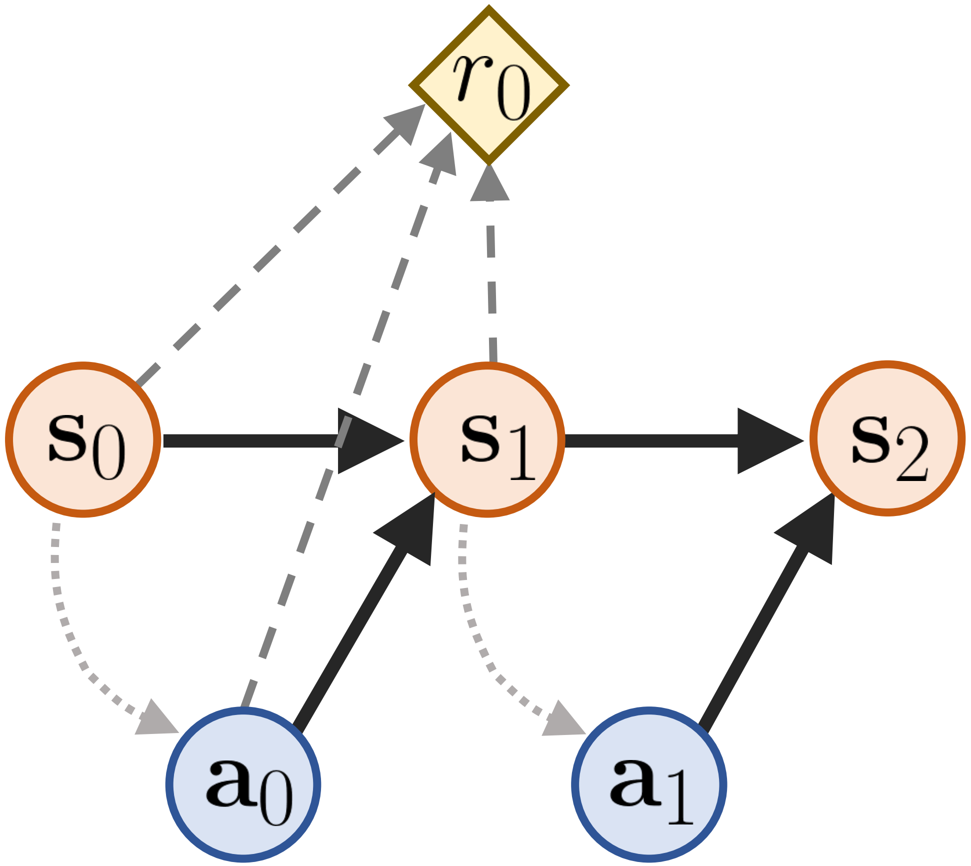
MDP



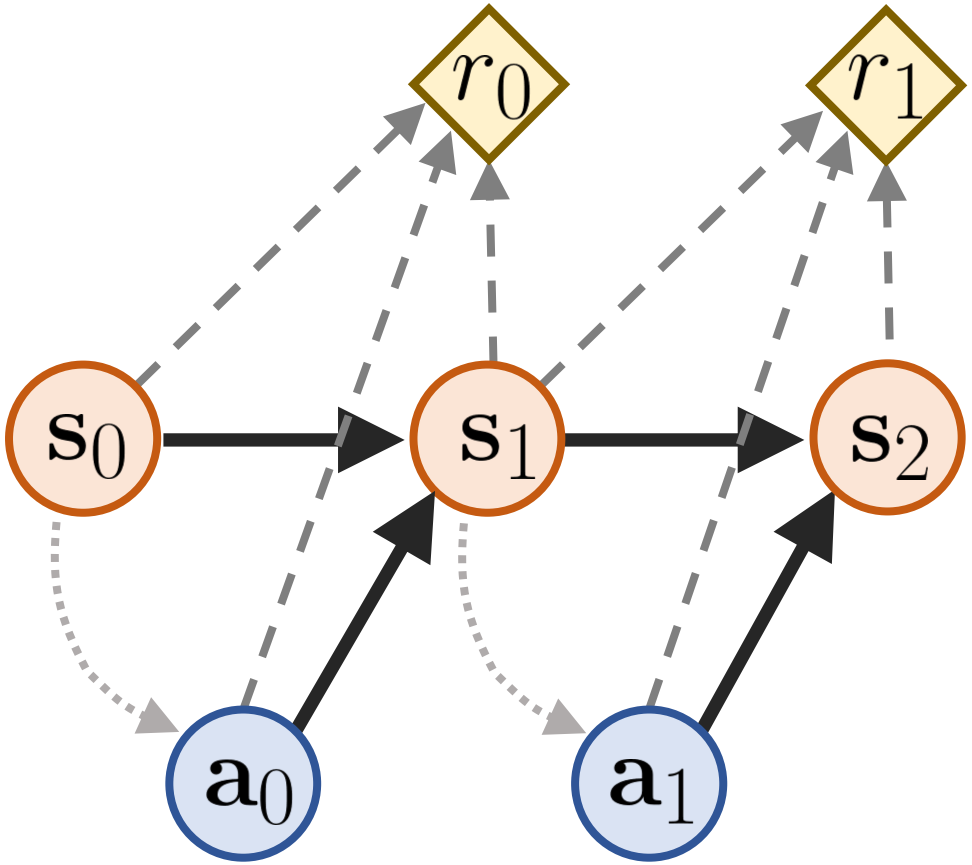
MDP



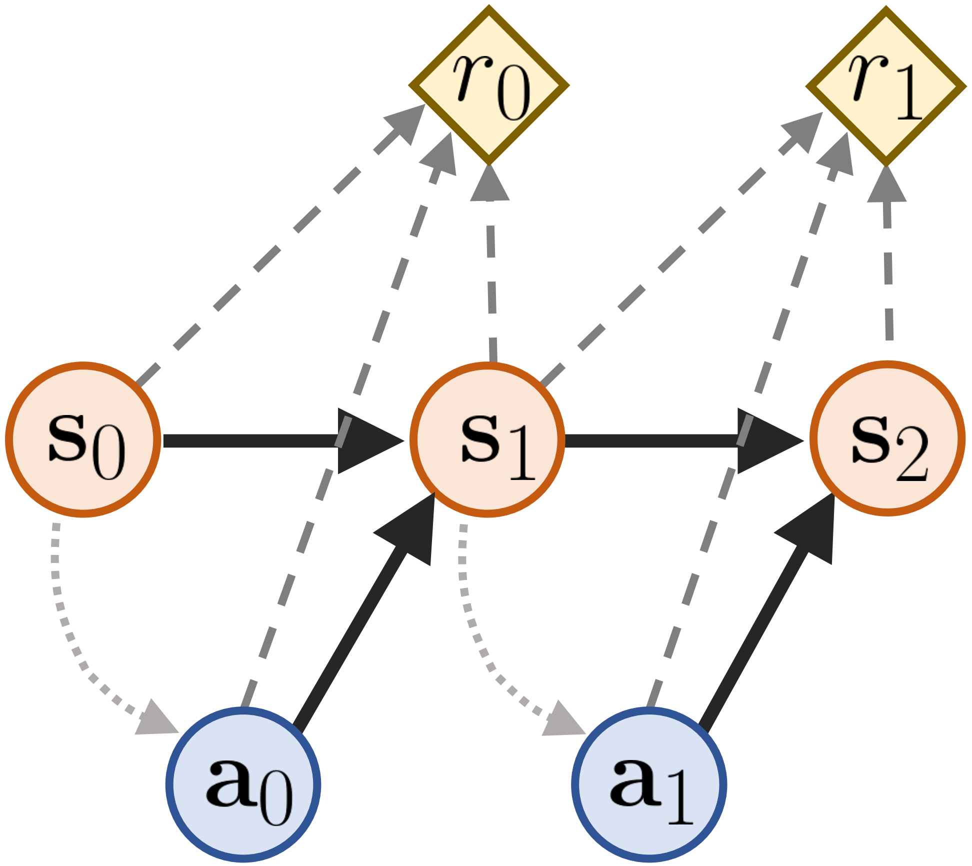
MDP



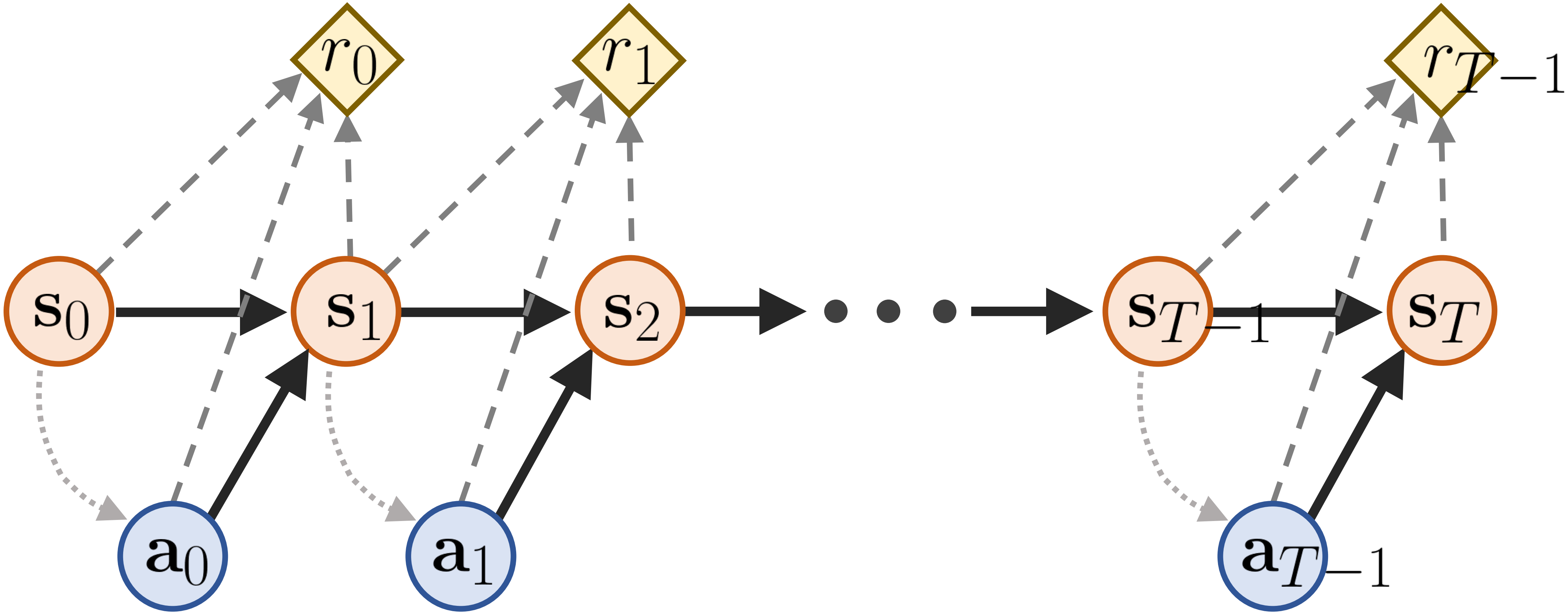
MDP



MDP

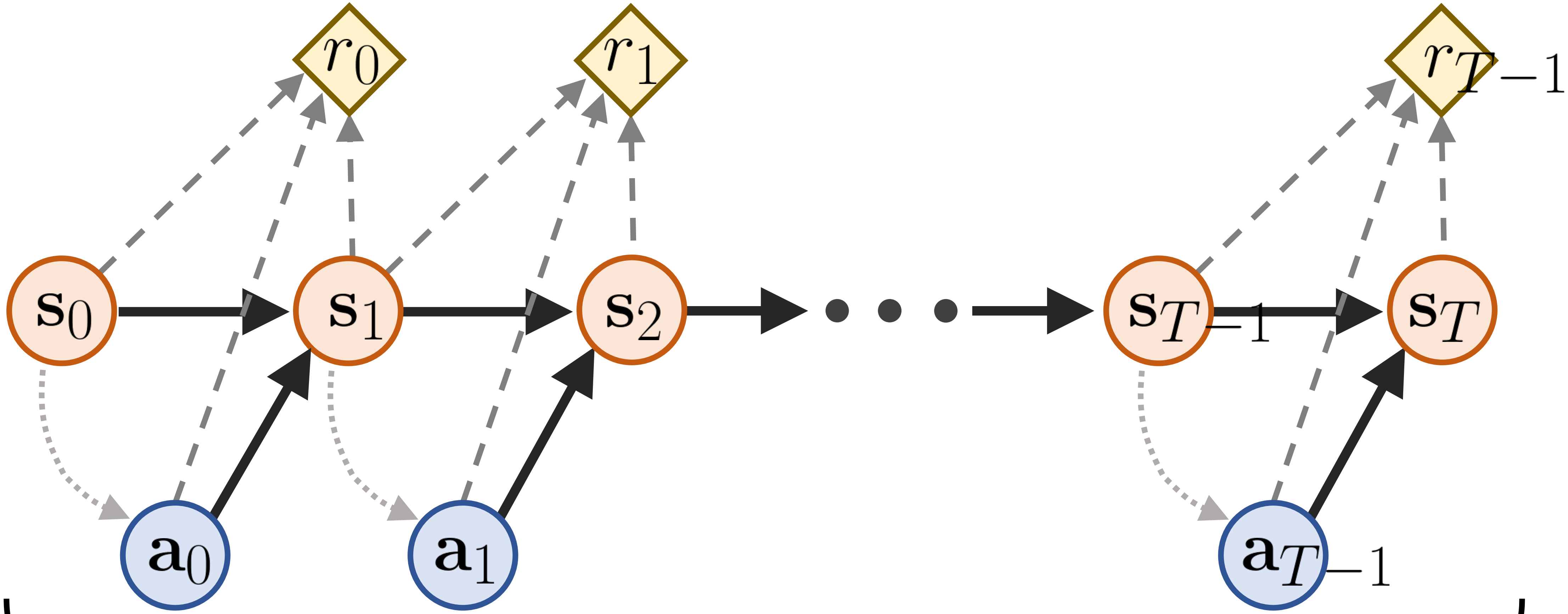


MDP



r_0
r_1
r_{T-1}
$\mathbf{s}_0$
$\mathbf{s}_1$
$\mathbf{s}_2$
$\mathbf{s}_{T-1}$
$\mathbf{s}_T$
$\mathbf{a}_0$
$\mathbf{a}_1$
$\mathbf{a}_{T-1}$

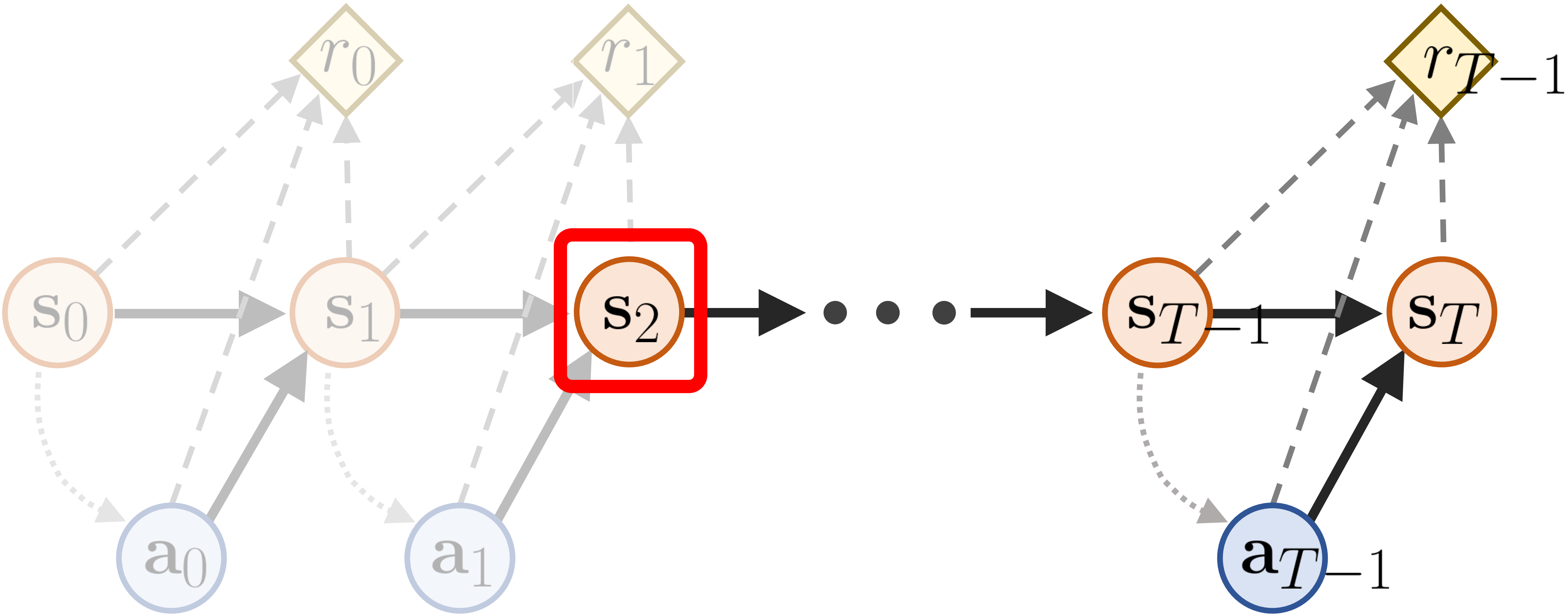
MDP



$$\max \boxed{\sum_{t=0}^{T-1} \gamma^t r_t}$$

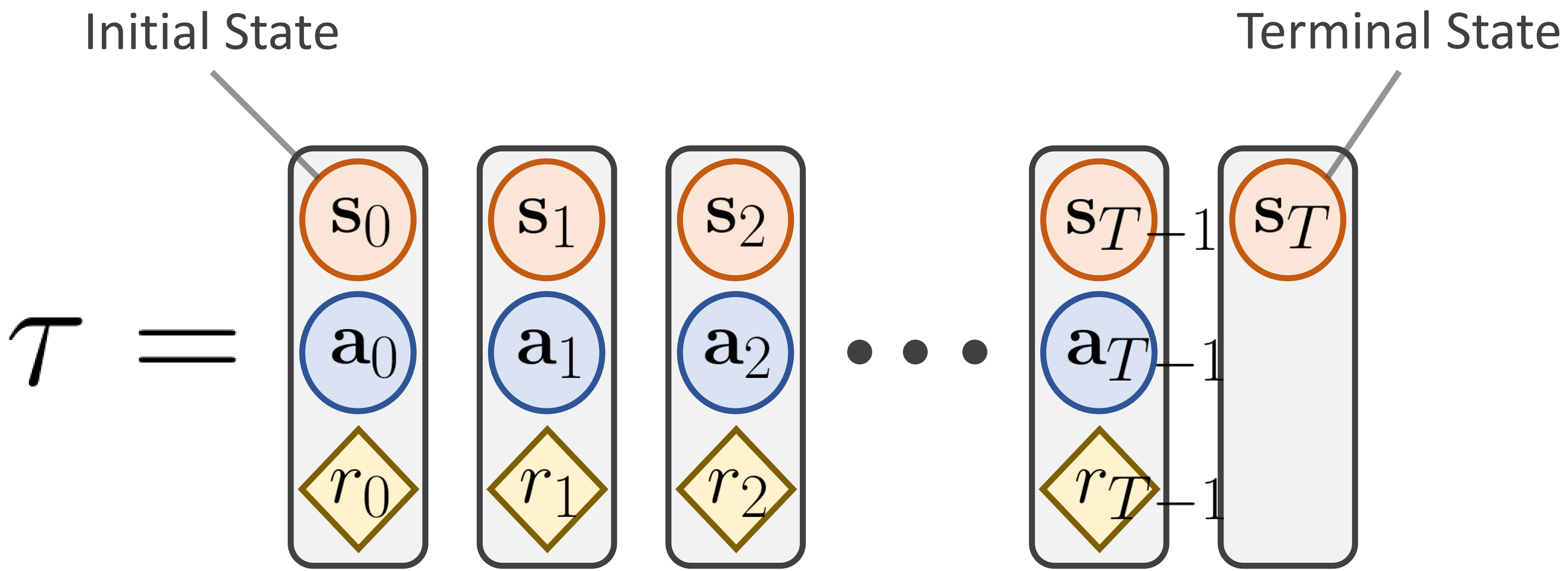
return

MDP



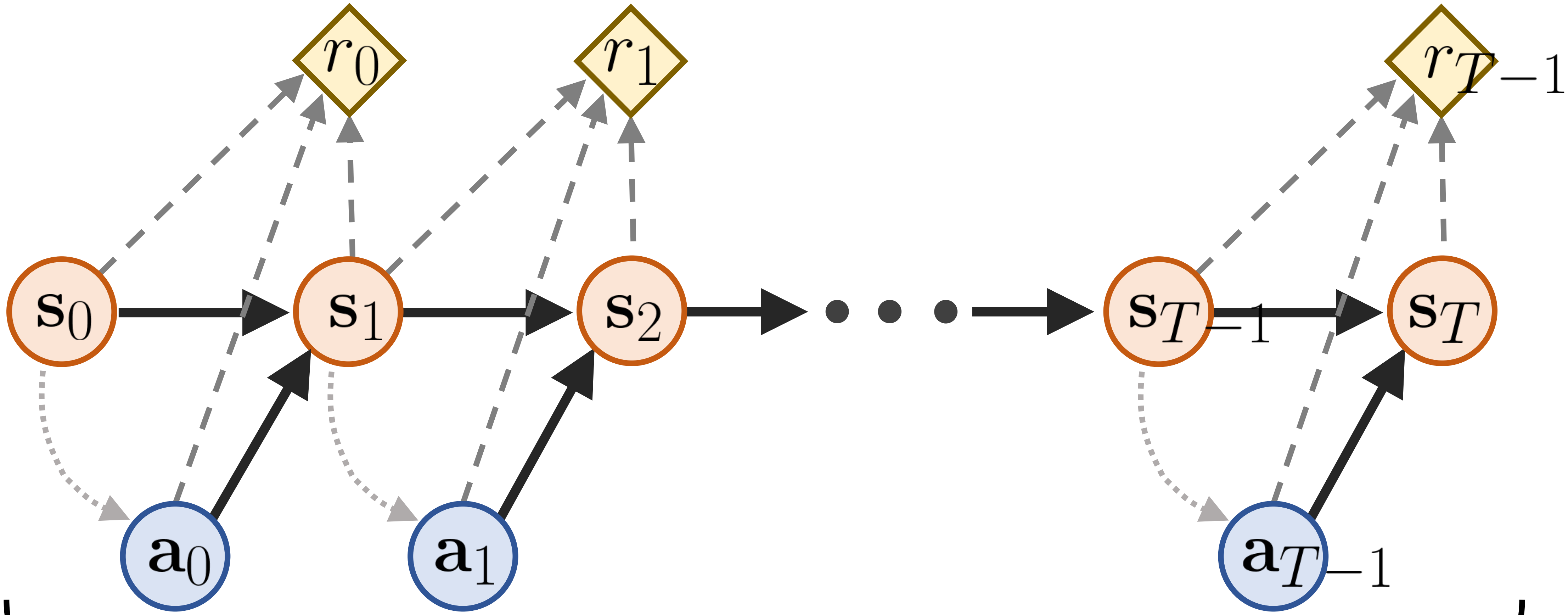
Markov property: $p(\mathbf{s}'|\mathbf{s},\mathbf{a})$

Trajectory



Episode: One trajectory/rollout.

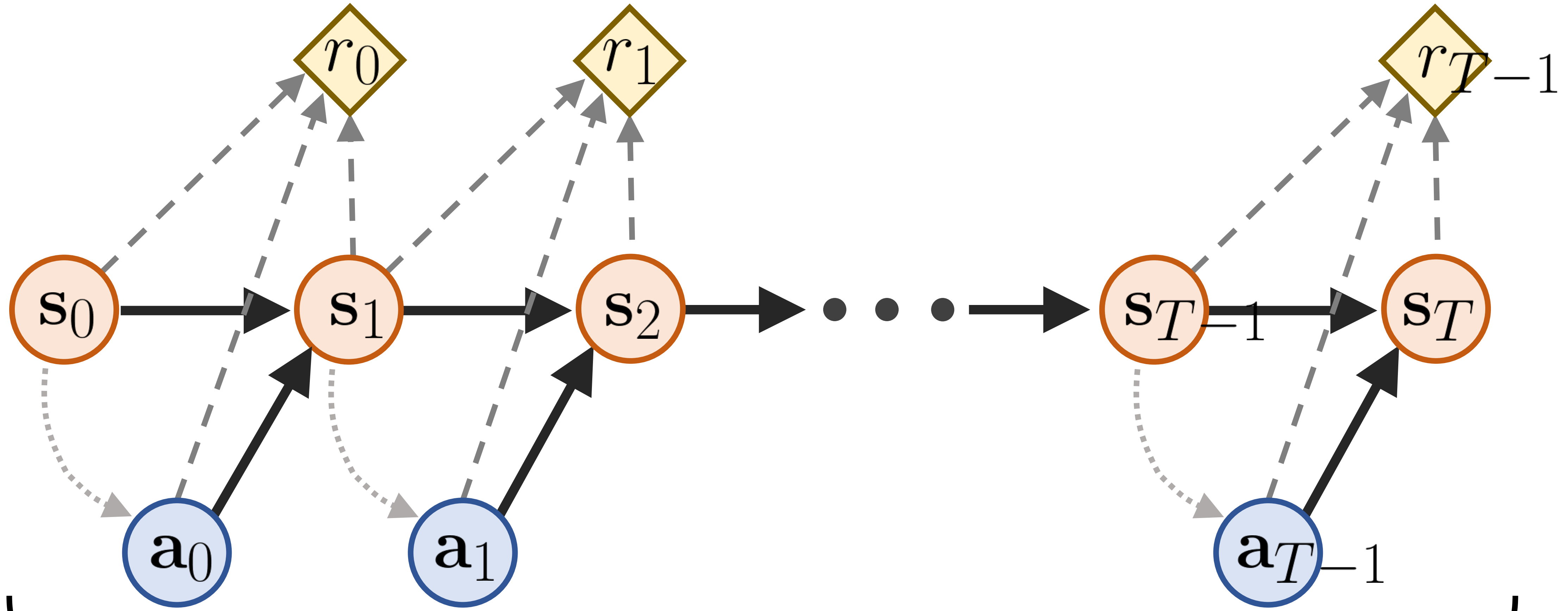
MDP



$$\max \sum_{t=0}^{\boxed{T}-1} \gamma^t r_t$$

$T \in [1, \infty]$

MDP



$$\underbrace{\qquad\qquad}_{\max \sum_{t=0}^{T-1} \boxed{\gamma^t} r_t}$$

$\gamma \in [0, 1]$

Discount Factor

$$\underline{J(\pi)} = \mathbb{E}_{\tau \sim p(\tau|\pi)} \left[\sum_{t=0}^{T-1} \gamma^t r_t \right]$$

Discount Factor

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Discount Factor

$$J(\pi) = \mathbb{E}_{\tau \sim p(\tau|\pi)} \left[\sum_{t=0}^{T-1} \gamma^t r_t \right]$$

$$R = r_0 + \gamma r_1 + \gamma^2 r_2 + \ldots + \gamma^{T-1} r_{T-1}$$

γ = 1: maximize rewards at all timesteps equally

Discount Factor

$$J(\pi) = \mathbb{E}_{\tau \sim p(\tau|\pi)} \left[\sum_{t=0}^{T-1} \gamma^t r_t \right]$$

$$R = r_0 + r_1 + r_2 + \ldots + r_{T-1}$$

γ = 1: maximize rewards at all timesteps equally

Discount Factor

$$J(\pi) = \mathbb{E}_{\tau \sim p(\tau|\pi)} \left[\sum_{t=0}^{T-1} \gamma^t r_t \right]$$

$$R = r_0 + \gamma r_1 + \gamma^2 r_2 + \ldots + \gamma^{T-1} r_{T-1}$$

$\gamma = 0$: only maximize reward at first timestep

Discount Factor

$$J(\pi) = \mathbb{E}_{\tau \sim p(\tau|\pi)} \left[\sum_{t=0}^{T-1} \gamma^t r_t \right]$$

$$R = r_0$$

$\gamma = 0$: only maximize reward at first timestep

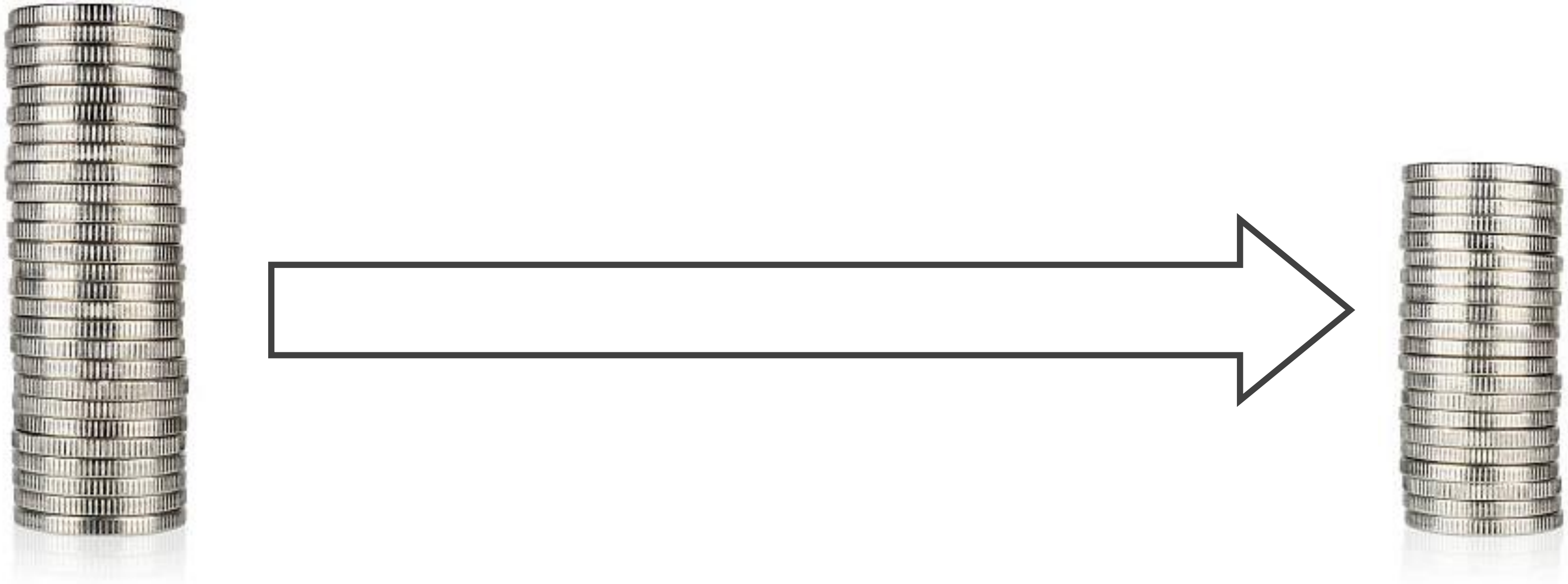
Discount Factor

$$J(\pi) = \mathbb{E}_{\tau \sim p(\tau|\pi)} \left[\sum_{t=0}^{T-1} \gamma^t r_t \right]$$

$$R = r_0 + 0.5 r_1 + 0.5^2 r_2 + \ldots + 0.5^{T-1} r_{T-1}$$

$\gamma = 0.5$

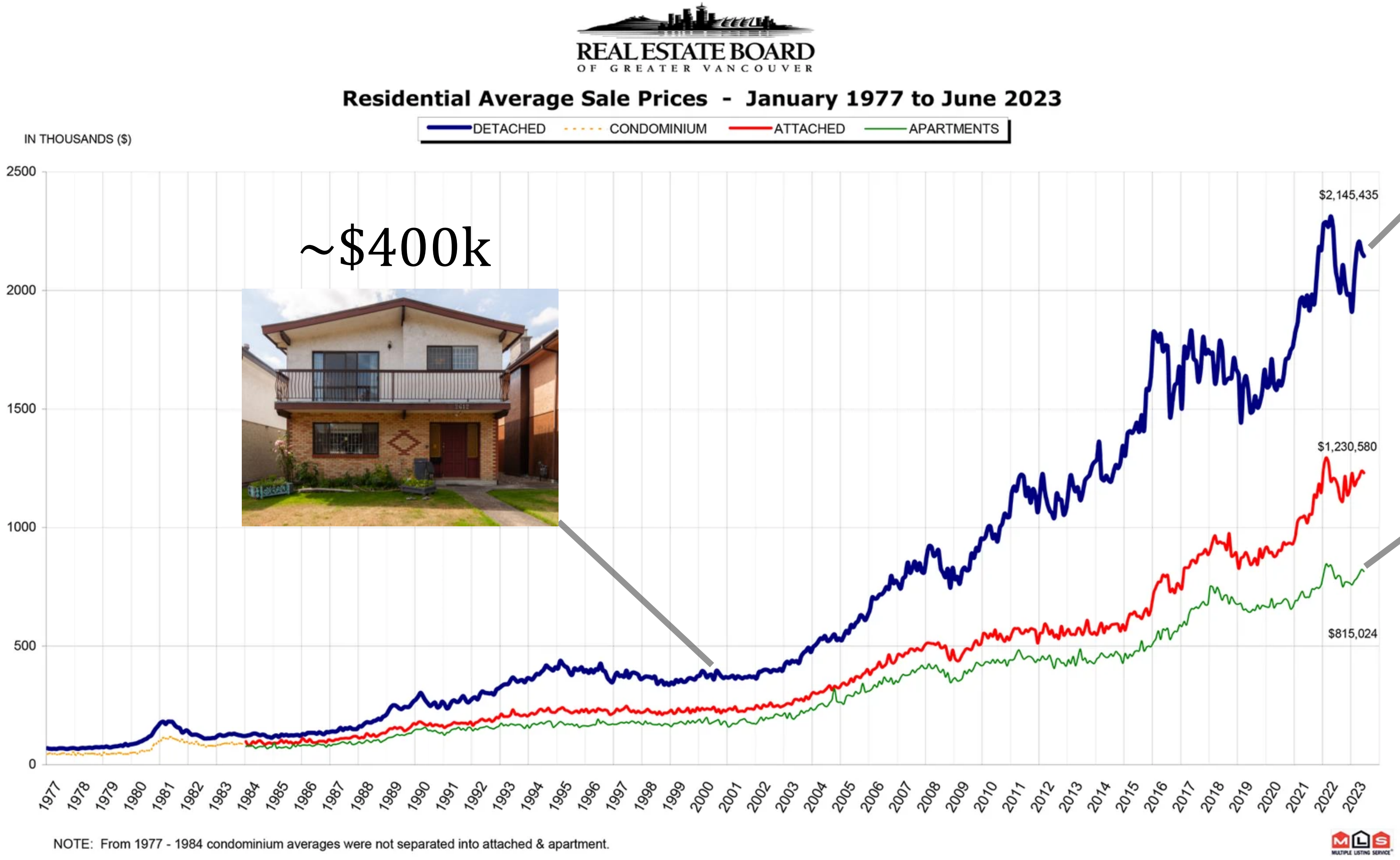
Inflation



Now

1 Year

Inflation



~$400k

~$2 mil

~$600k

Discount Factor

$$J(\pi) = \mathbb{E}_{\tau \sim p(\tau|\pi)} \left[\sum_{t=0}^{T-1} \gamma^t r_t \right]$$

$$R = r_0 + \gamma r_1 + \gamma^2 r_2 + \ldots + \gamma^{T-1} r_{T-1}$$

Large γ : maximize long term rewards (far-sighted)

Small γ : maximize short term rewards (short-sighted / greedy)

Discount Factor

$$J(\pi) = \mathbb{E}_{\tau \sim p(\tau|\pi)} \left[\sum_{t=0}^{T-1} \gamma^t r_t \right]$$

$$R = r_0 + \gamma r_1 + \gamma^2 r_2 + \ldots + \gamma^{T-1} r_{T-1}$$

$\gamma = 1$: maximize rewards at all timesteps equally

$T \neq \infty$

Expectation undefined if $T = \infty$

Discount Factor

$$J(\pi) = \mathbb{E}_{\tau \sim p(\tau|\pi)} \left[\sum_{t=0}^{T-1} \gamma^t r_t \right]$$

$$R = r_0 + \gamma r_1 + \gamma^2 r_2 + \ldots + \gamma^{T-1} r_{T-1}$$

If $T = \infty$,

Then $\gamma < 1$ for expectation to be defined.

Discount Factor

- Geometric series with ratio $\gamma < 1$
- $r \in [r_{\min}, r_{\max}], \quad r \geq 0$

$$R = r_0 + \gamma r_1 + \gamma^2 r_2 + \ldots + \gamma^{T-1} r_{T-1}$$

Return Bounds

- Geometric series with ratio $\gamma < 1$
- $r \in [r_{\min}, r_{\max}], \quad r \geq 0$

Return Bounds

- Geometric series with ratio $\gamma < 1$
- $r \in [r_{\min}, r_{\max}], \;\; r \geq 0$

$$R = \sum_{t=0}^{T-1} \gamma^t \, r_t$$

Return Bounds

- Geometric series with ratio $\gamma < 1$
- $r \in [r_{\min}, r_{\max}], \;\; r \geq 0$

$$R = \sum_{t=0}^{T-1} \gamma^t \, r_t \leq \sum_{t=0}^{T-1} \gamma^t \, r_{\max}$$

Return Bounds

- Geometric series with ratio $\gamma < 1$
- $r \in [r_{\min}, r_{\max}], \;\; r \geq 0$

$$R = \sum_{t=0}^{T-1} \gamma^t \, r_t \leq \sum_{t=0}^{T-1} \gamma^t \, r_{\max}$$

$$\leq \sum_{t=0}^{\infty} \gamma^t \, r_{\max}$$

Return Bounds

- Geometric series with ratio $\gamma < 1$
- $r \in [r_{\min}, r_{\max}], \quad r \geq 0$

$$R = \sum_{t=0}^{T-1} \gamma^t \, r_t \leq \sum_{t=0}^{T-1} \gamma^t \, r_{\max}$$

$$\leq \sum_{t=0}^{\infty} \gamma^t \, r_{\max} = r_{\max} \underline{\sum_{t=0}^{\infty} \gamma^t}$$

Return Bounds

- Geometric series with ratio $\gamma < 1$
- $r \in [r_{\min}, r_{\max}], \quad r \geq 0$

$$R = \sum_{t=0}^{T-1} \gamma^t \, r_t \leq \sum_{t=0}^{T-1} \gamma^t \, r_{\max}$$

$$\leq \sum_{t=0}^{\infty} \gamma^t \, r_{\max} = r_{\max} \sum_{t=0}^{\infty} \gamma^t$$

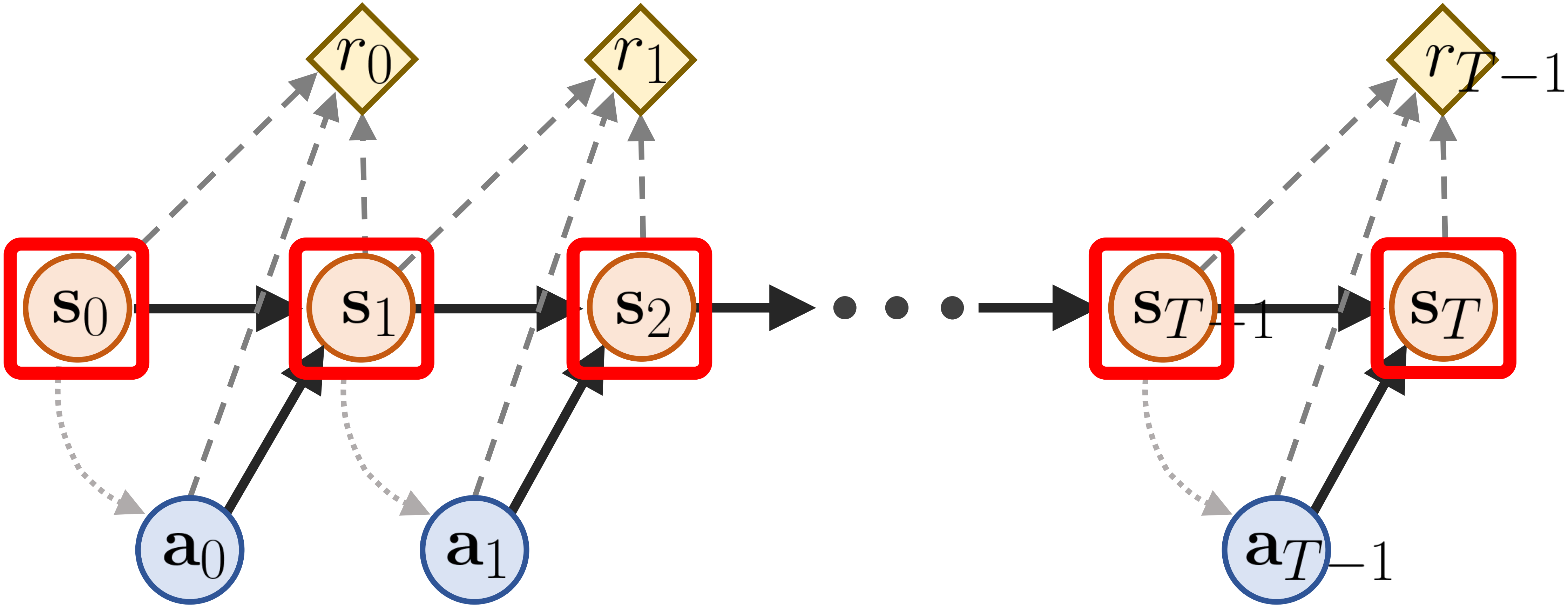
$$\boxed{\leq r_{\max} \frac{1}{1-\gamma}}$$

Return Bounds

$$R_{\min} = \frac{1}{1-\gamma} r_{\min} \qquad R_{\max} = \frac{1}{1-\gamma} r_{\max}$$

$$R_{\min} \leq R \leq R_{\max}$$

MDP



r_0
r_1
r_{T-1}
$\mathbf{s}_0$
$\mathbf{s}_1$
$\mathbf{s}_2$
$\mathbf{s}_{T-1}$
$\mathbf{s}_T$
$\mathbf{a}_0$
$\mathbf{a}_1$
$\mathbf{a}_{T-1}$

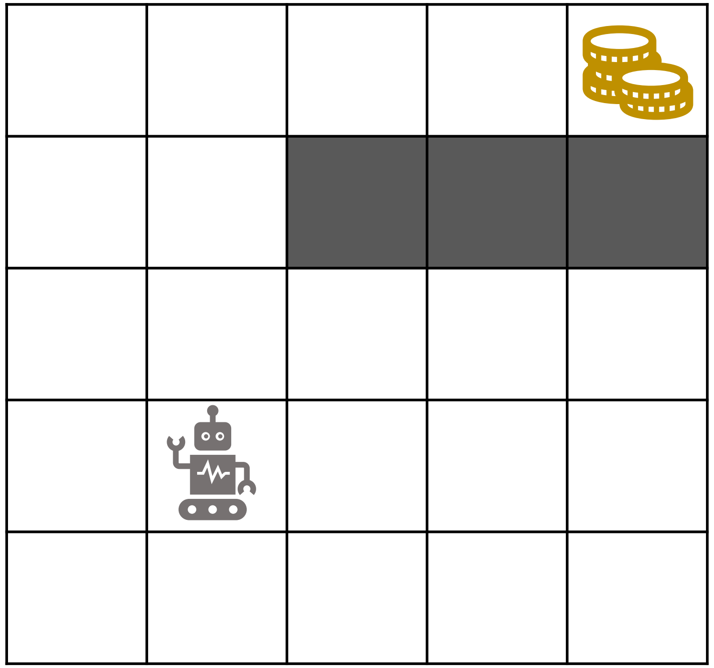
State Spaces

Discrete

$$\mathbf{s} \in \{\mathbf{s}^0, \mathbf{s}^1, \mathbf{s}^2, \ldots, \mathbf{s}^n\}$$

Continuous

$$\mathbf{s} \in \mathbb{R}^n$$



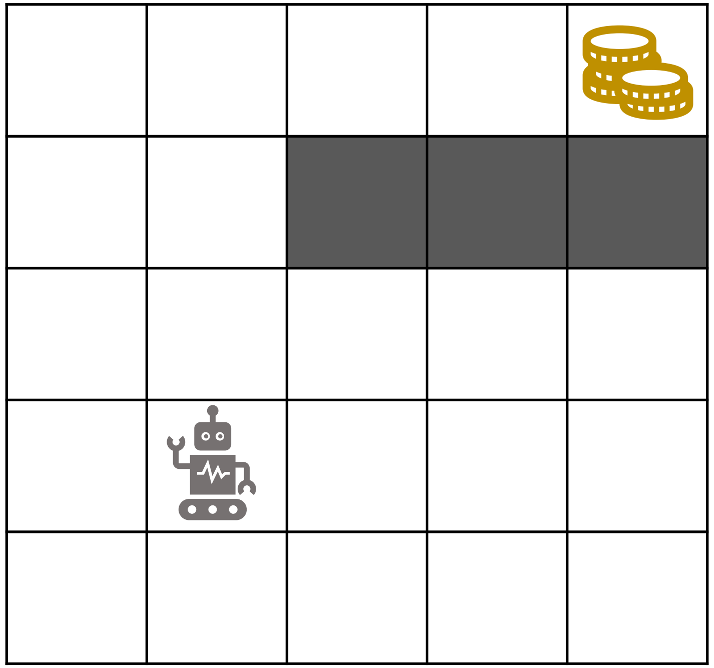
State Spaces

Discrete

$$\mathbf{s} \in \{\mathbf{s}^0, \mathbf{s}^1, \mathbf{s}^2, \dots, \mathbf{s}^n\}$$

Continuous

$$\mathbf{s} \in \mathbb{R}^n$$



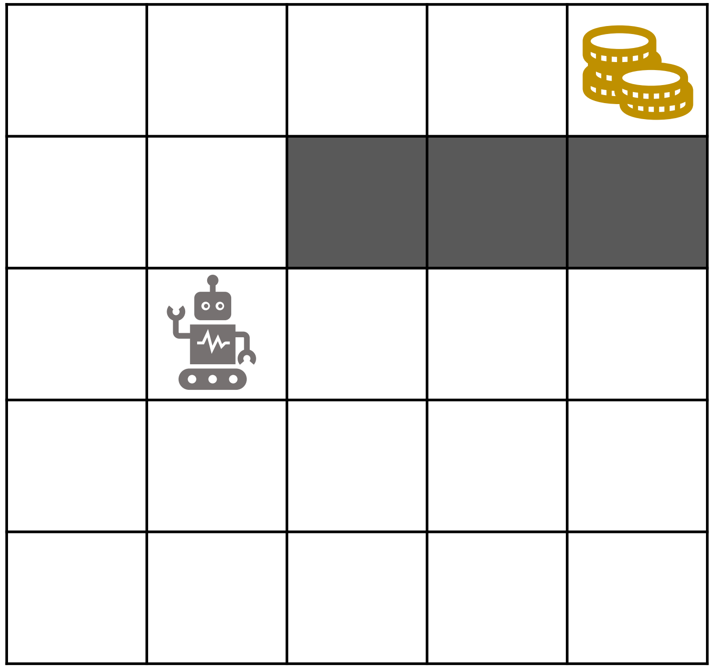
State Spaces

Discrete

$$\mathbf{s} \in \{\mathbf{s}^0, \mathbf{s}^1, \mathbf{s}^2, ..., \mathbf{s}^n\}$$

Continuous

$$\mathbf{s} \in \mathbb{R}^n$$



State Spaces

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Continuous

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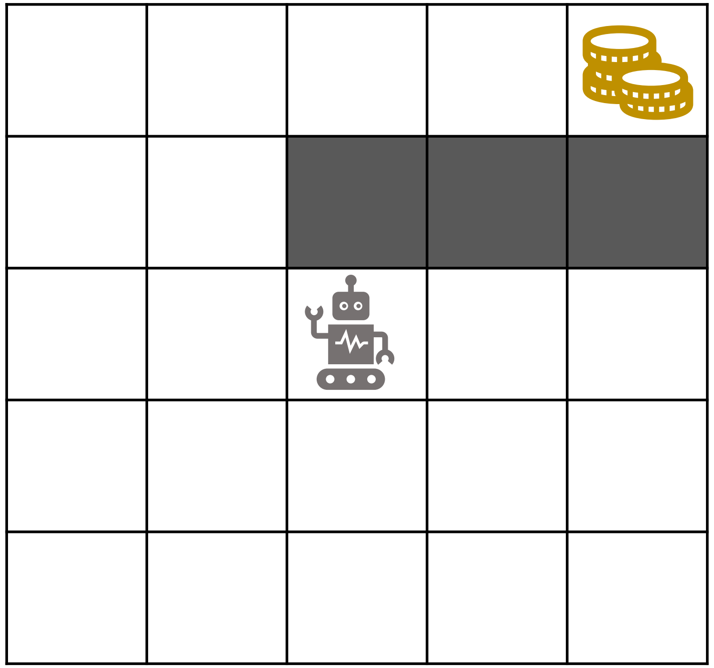

State Spaces

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Continuous

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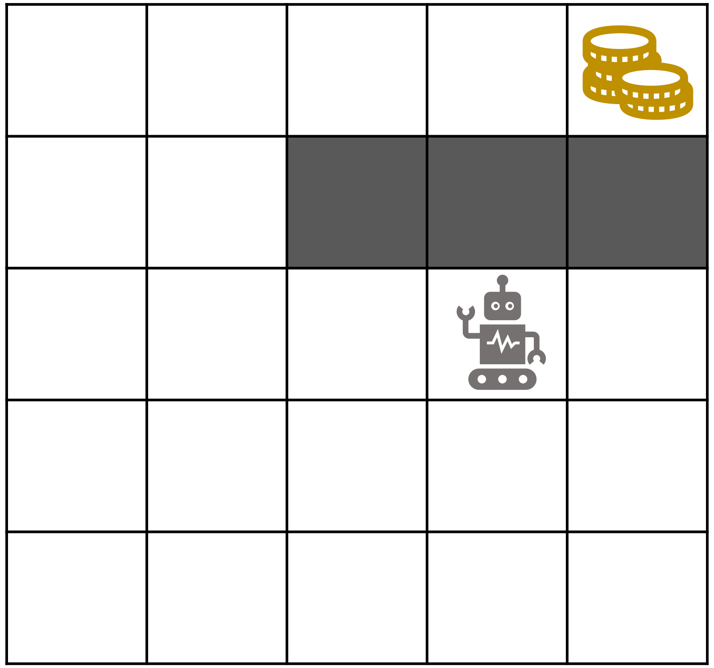

State Spaces

Discrete

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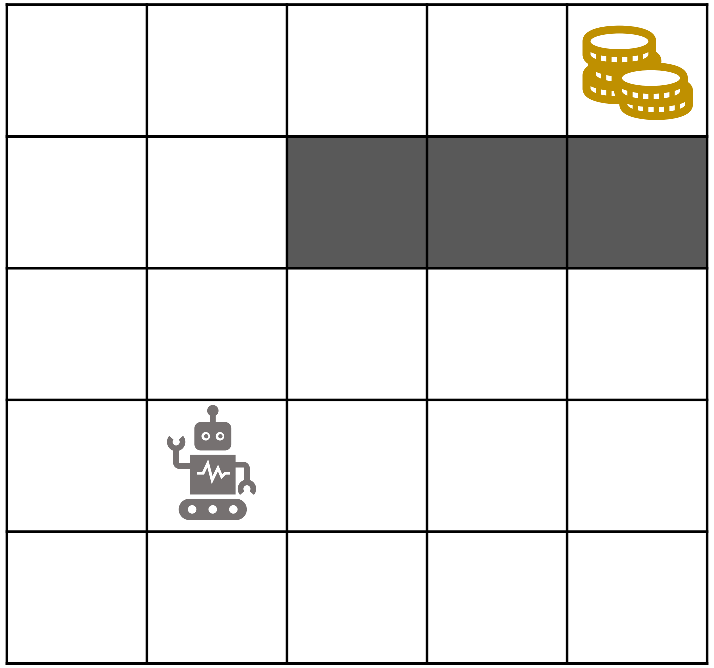
Continuous

$$\mathbf{s} \in \mathbb{R}^n$$



State Spaces

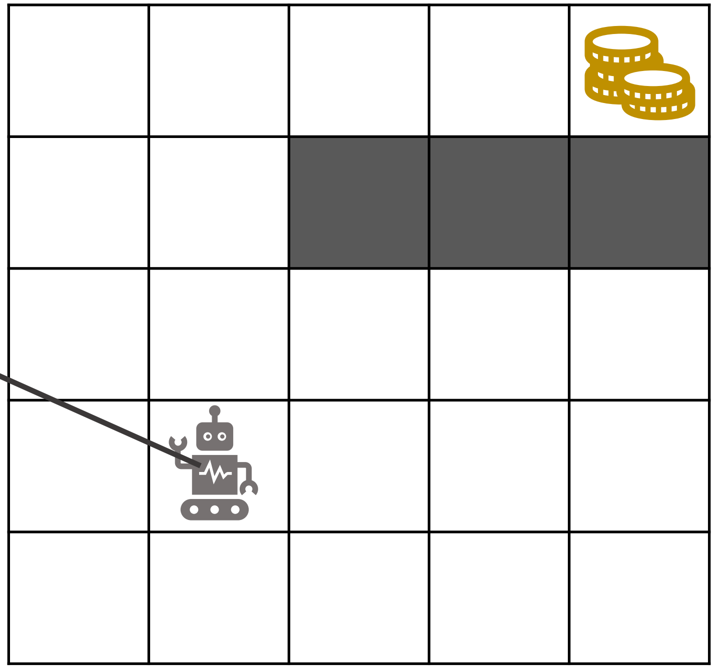
Discrete

$$\mathbf{s} \in \{\mathbf{s}^0, \mathbf{s}^1, \mathbf{s}^2, ..., \mathbf{s}^n\}$$

Continuous

$$\mathbf{s} \in \mathbb{R}^n$$

$$\mathbf{s} = (x, y)$$



State Spaces

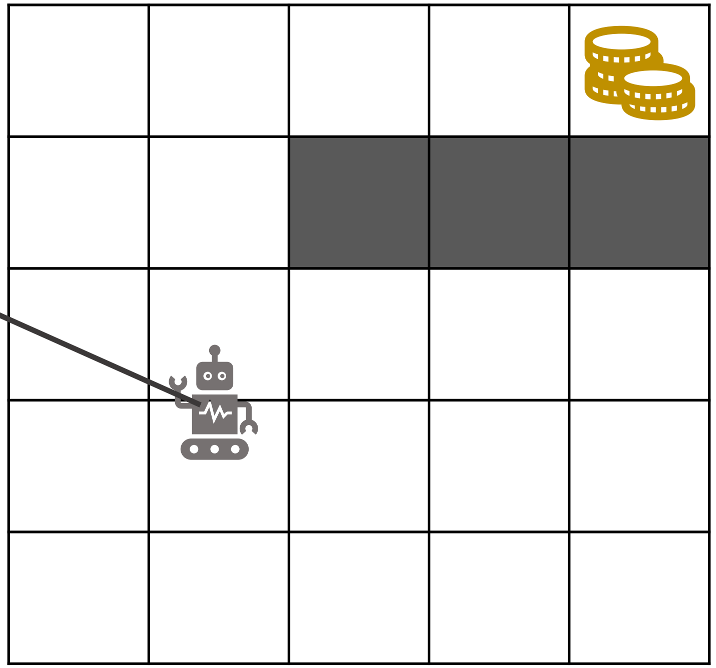
Discrete

$$\mathbf{s} \in \{\mathbf{s}^0, \mathbf{s}^1, \mathbf{s}^2, \ldots, \mathbf{s}^n\}$$

Continuous

$$\mathbf{s} \in \mathbb{R}^n$$

$$\mathbf{s} = (x, y)$$



State Spaces

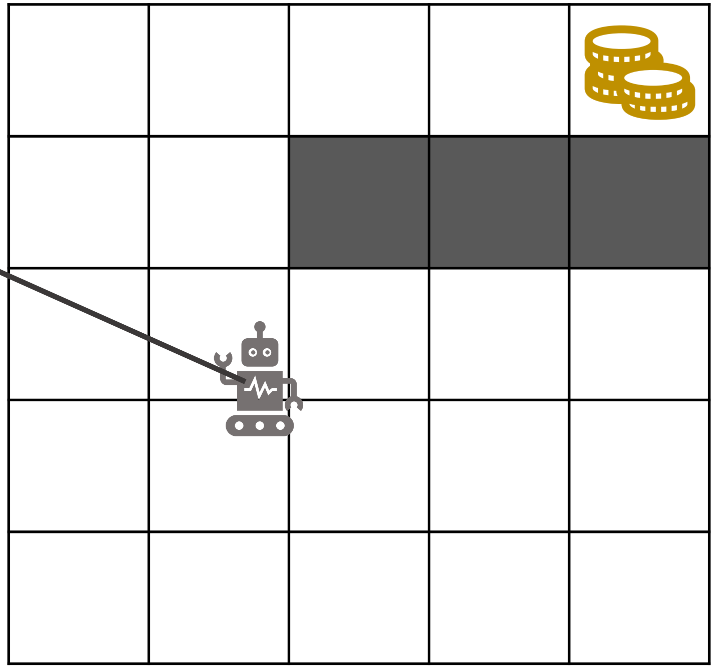
Discrete

$$\mathbf{s} \in \{\mathbf{s}^0, \mathbf{s}^1, \mathbf{s}^2, ..., \mathbf{s}^n\}$$

Continuous

$$\mathbf{s} \in \mathbb{R}^n$$

$$\mathbf{s} = (x, y)$$



State Spaces

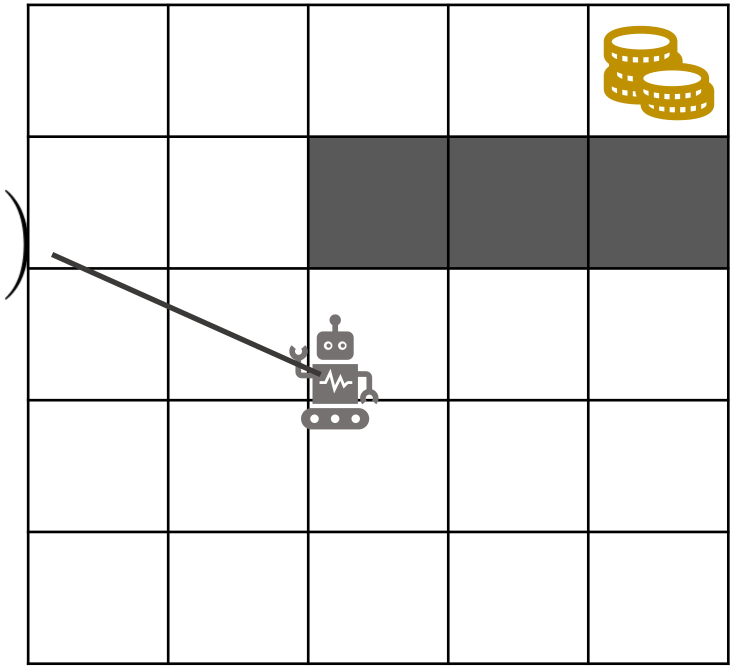
Discrete

$$\mathbf{s} \in \{\mathbf{s}^0, \mathbf{s}^1, \mathbf{s}^2, \ldots, \mathbf{s}^n\}$$

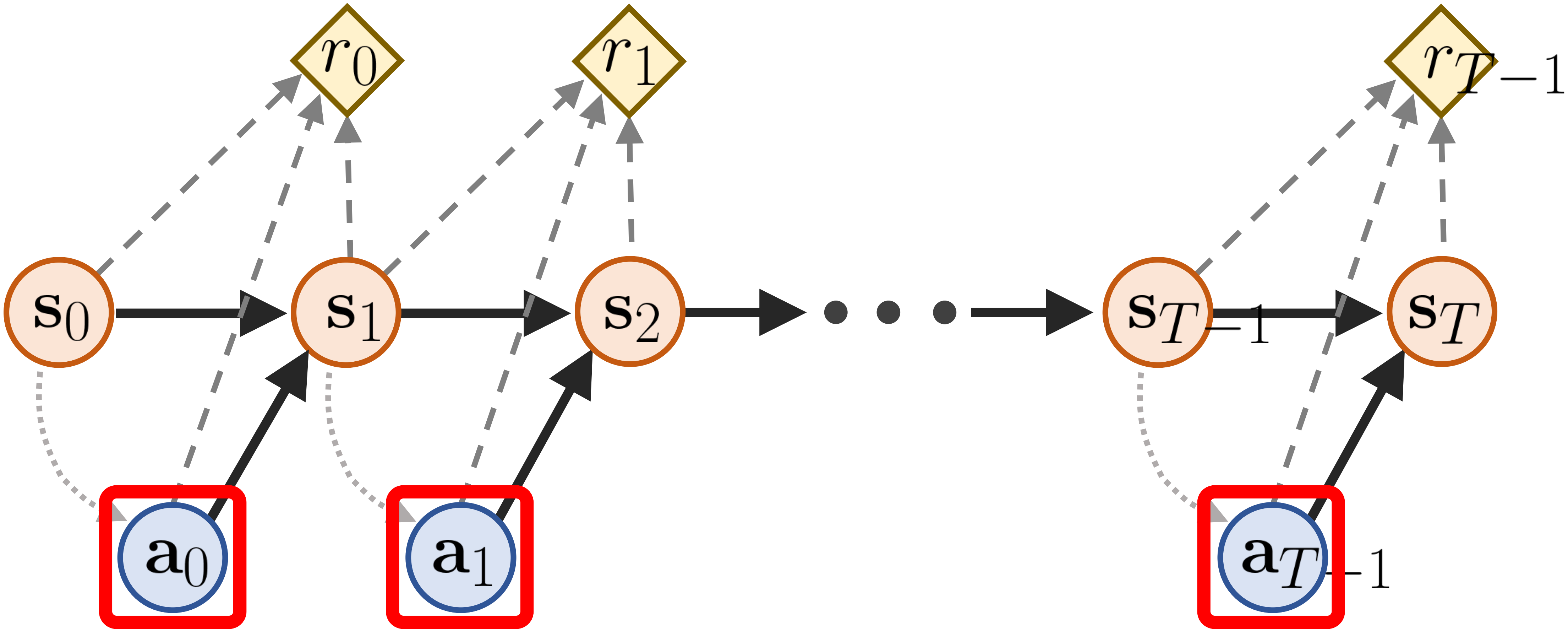
Continuous

$$\mathbf{s} \in \mathbb{R}^n$$

$$\mathbf{s} = (x, y)$$



MDP


r_0
r_1
r_{T-1}
$\mathbf{s}_0$
$\mathbf{s}_1$
$\mathbf{s}_2$
$\mathbf{s}_{T-1}$
$\mathbf{s}_T$
$\mathbf{a}_0$
$\mathbf{a}_1$
$\mathbf{a}_{T-1}$

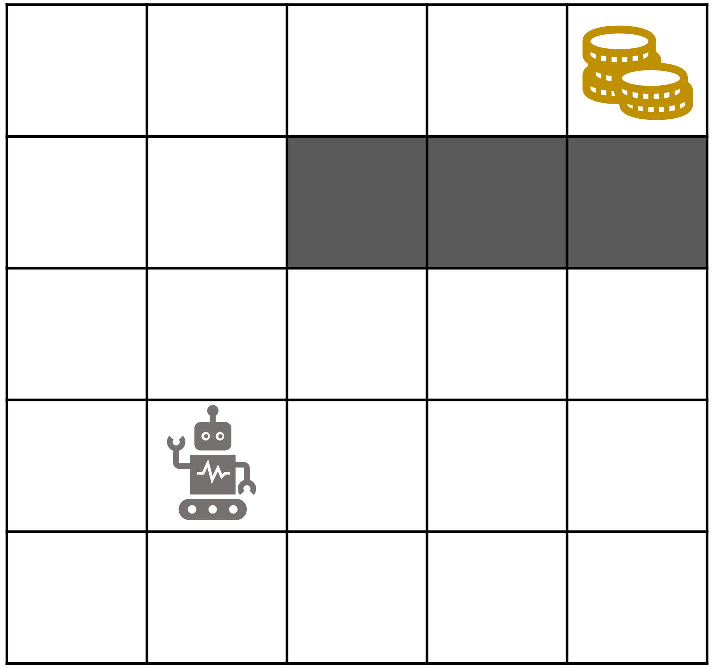
Action Spaces

Discrete

$$\mathbf{a} \in \{\mathbf{a}^0, \mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^m\}$$

Continuous

$$\mathbf{a} \in \mathbb{R}^m$$



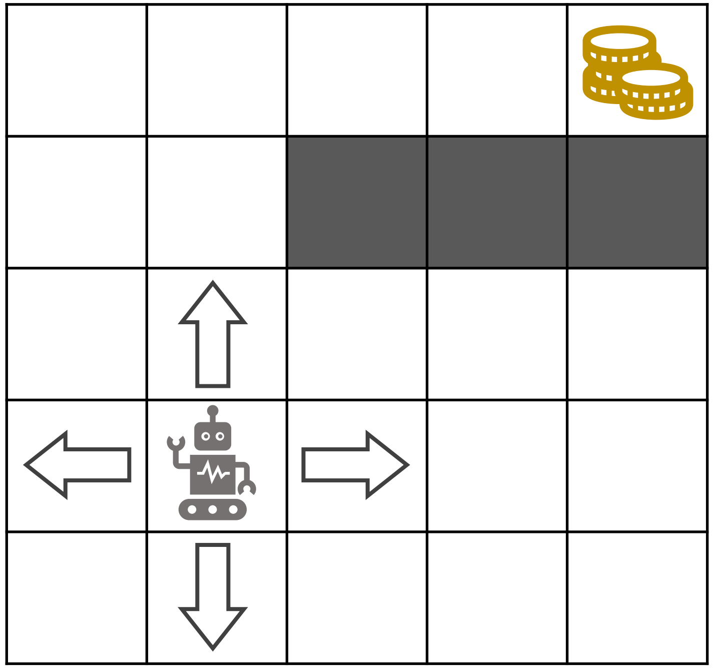
Action Spaces

Discrete

$$\mathbf{a} \in \{\mathbf{a}^0, \mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^m\}$$

Continuous

$$\mathbf{a} \in \mathbb{R}^m$$



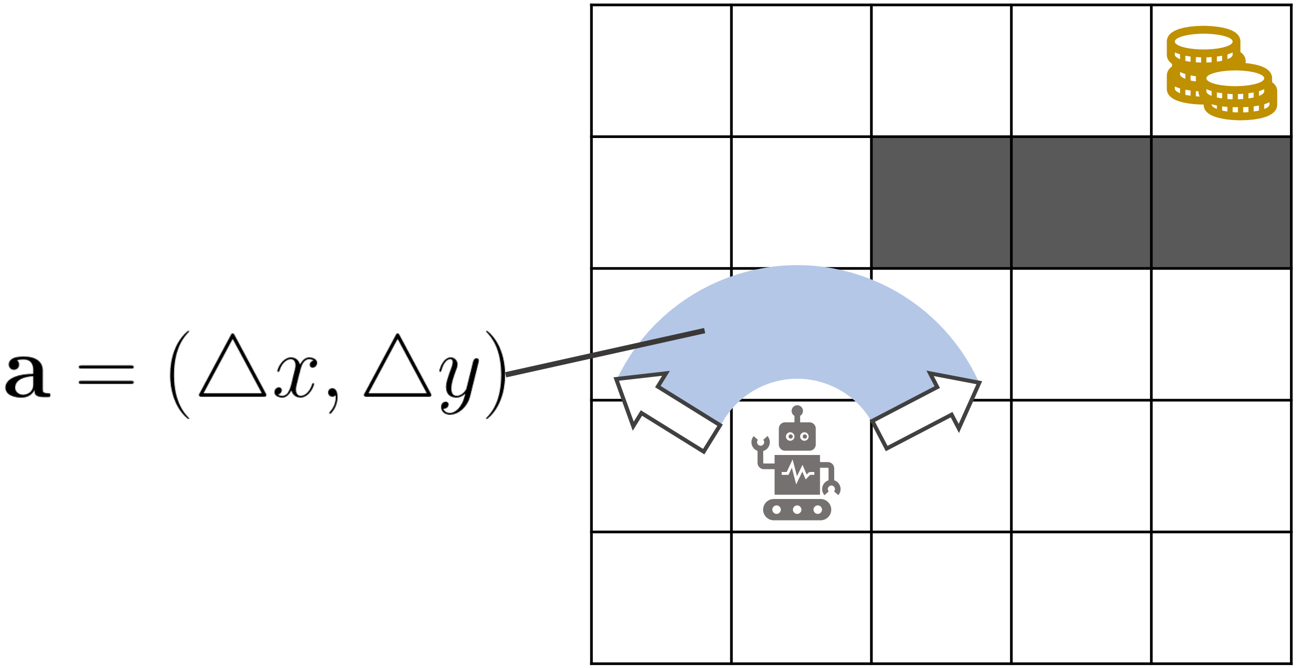
Action Spaces

Discrete

$$\mathbf{a} \in \{\mathbf{a}^0, \mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^m\}$$

Continuous

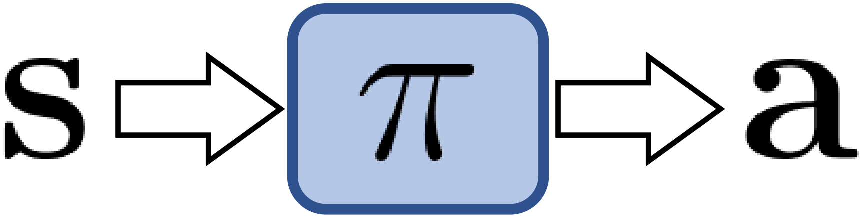
$$\mathbf{a} \in \mathbb{R}^m$$



Policies

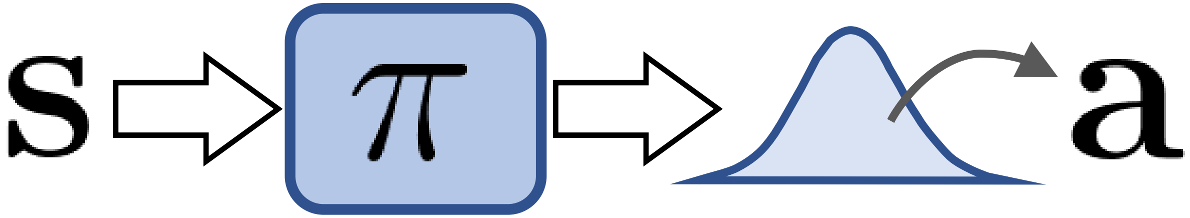
Deterministic

$$\mathbf{a} = \pi(\mathbf{s})$$



Stochastic

$$\pi(\mathbf{a}|\mathbf{s})$$



Deterministic Policy

$$\pi^{\text{det}}(\mathbf{s}) = \mathbf{a}^*$$

$$\pi(\mathbf{a}|\mathbf{s}) = \begin{cases} 1 & \text{if } \mathbf{a} = \pi^{\text{det}}(\mathbf{s}) \\ 0 & \text{otherwise} \end{cases}$$

Deterministic Policy

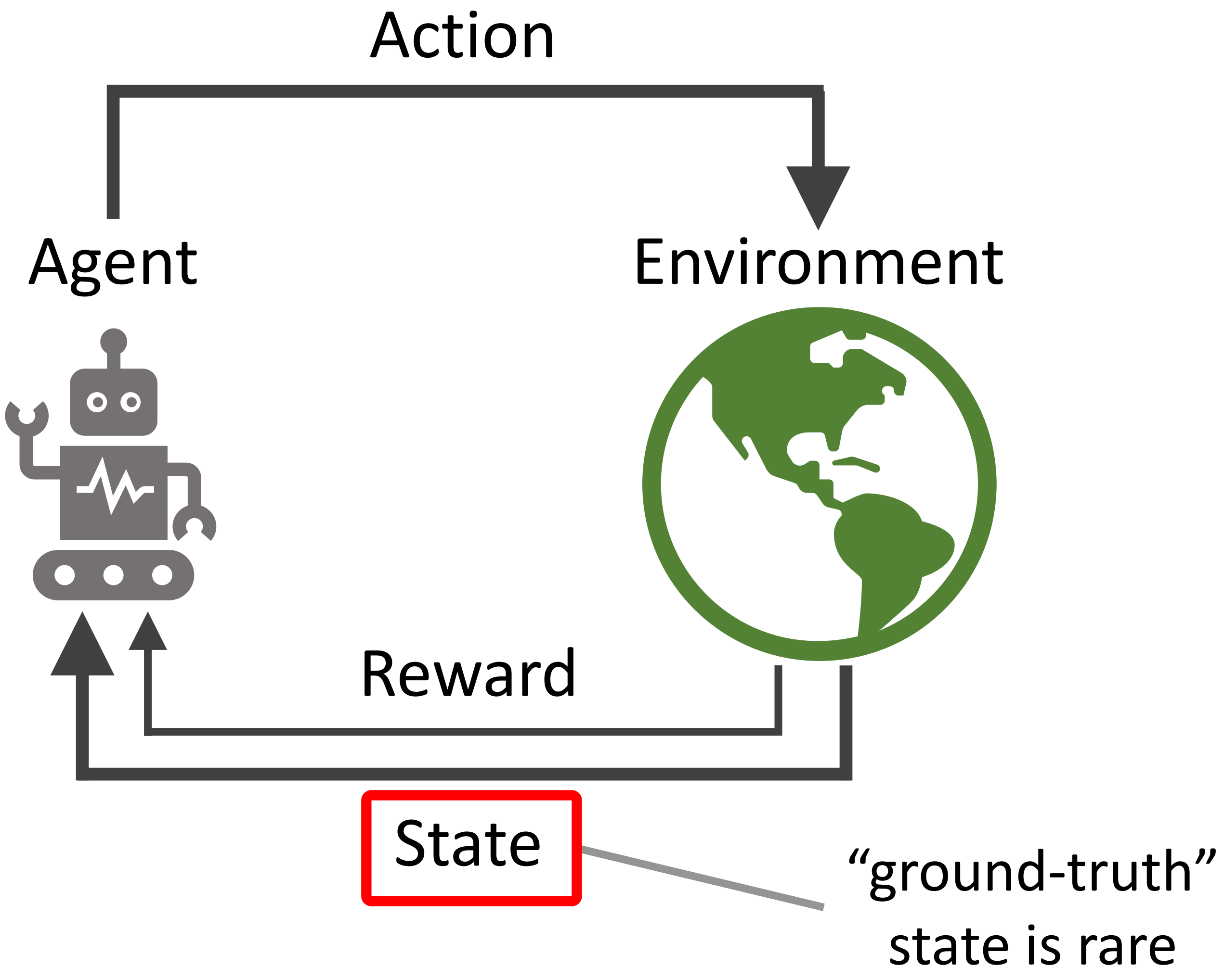
$$\pi(\mathbf{a}|\mathbf{s})$$

Deterministic Policy

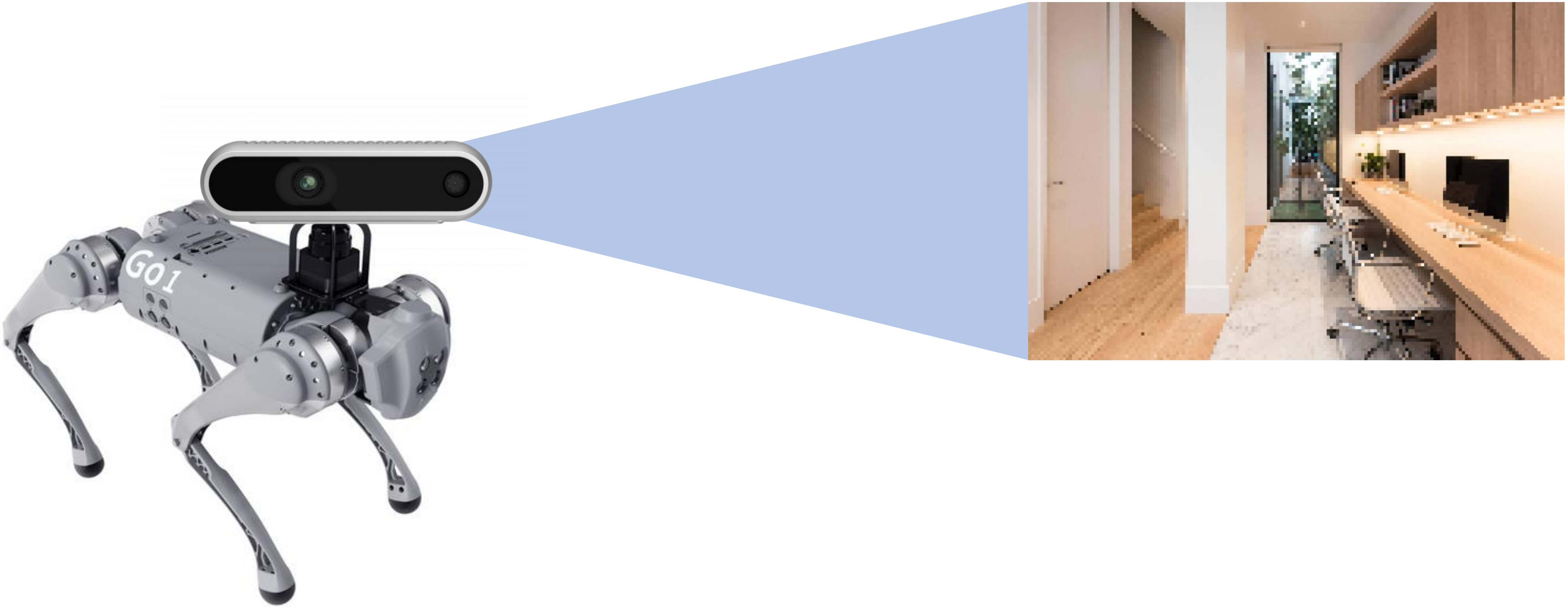
$$\mathbf{a}^* = \arg\max_{\mathbf{a}} \; \pi(\mathbf{a}|\mathbf{s})$$

$$\pi^{\mathrm{det}}(\mathbf{s}) = \mathbf{a}^*$$

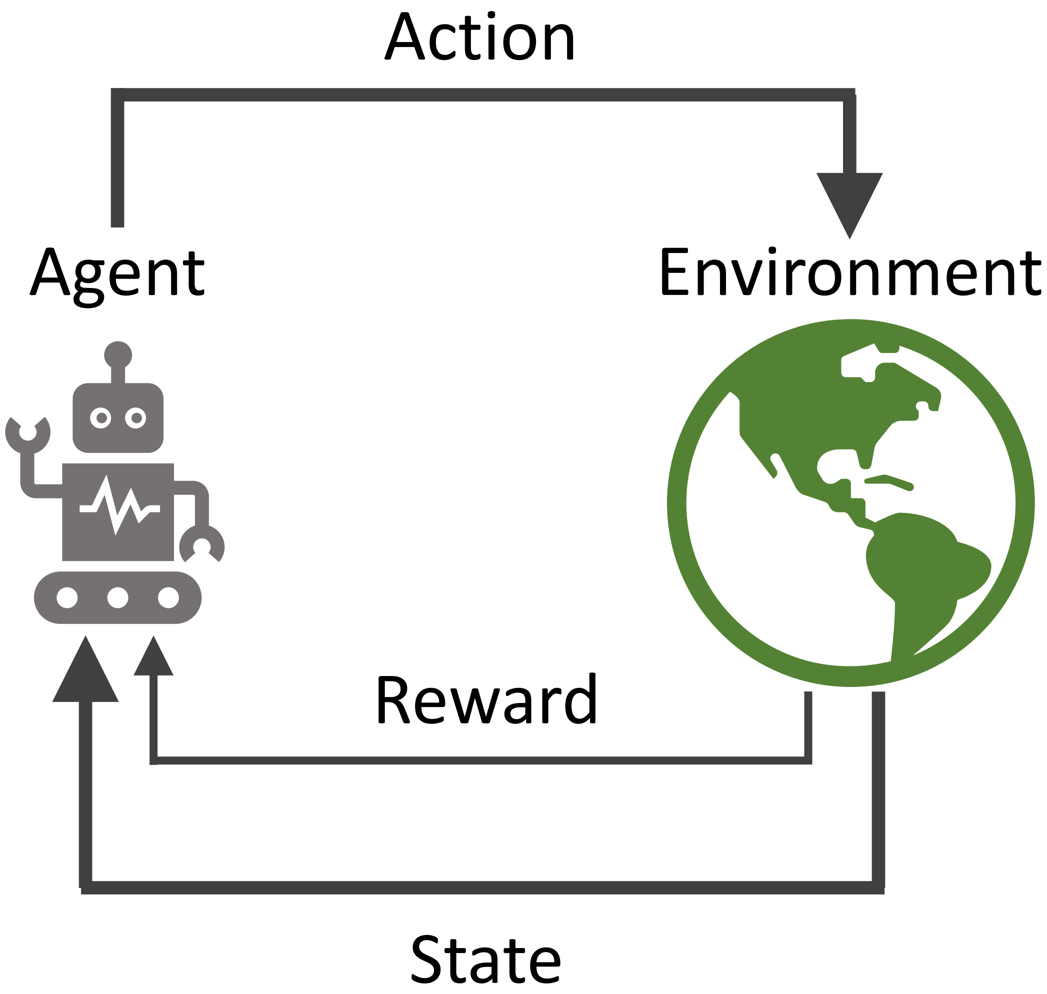
Agent-Environment Interface



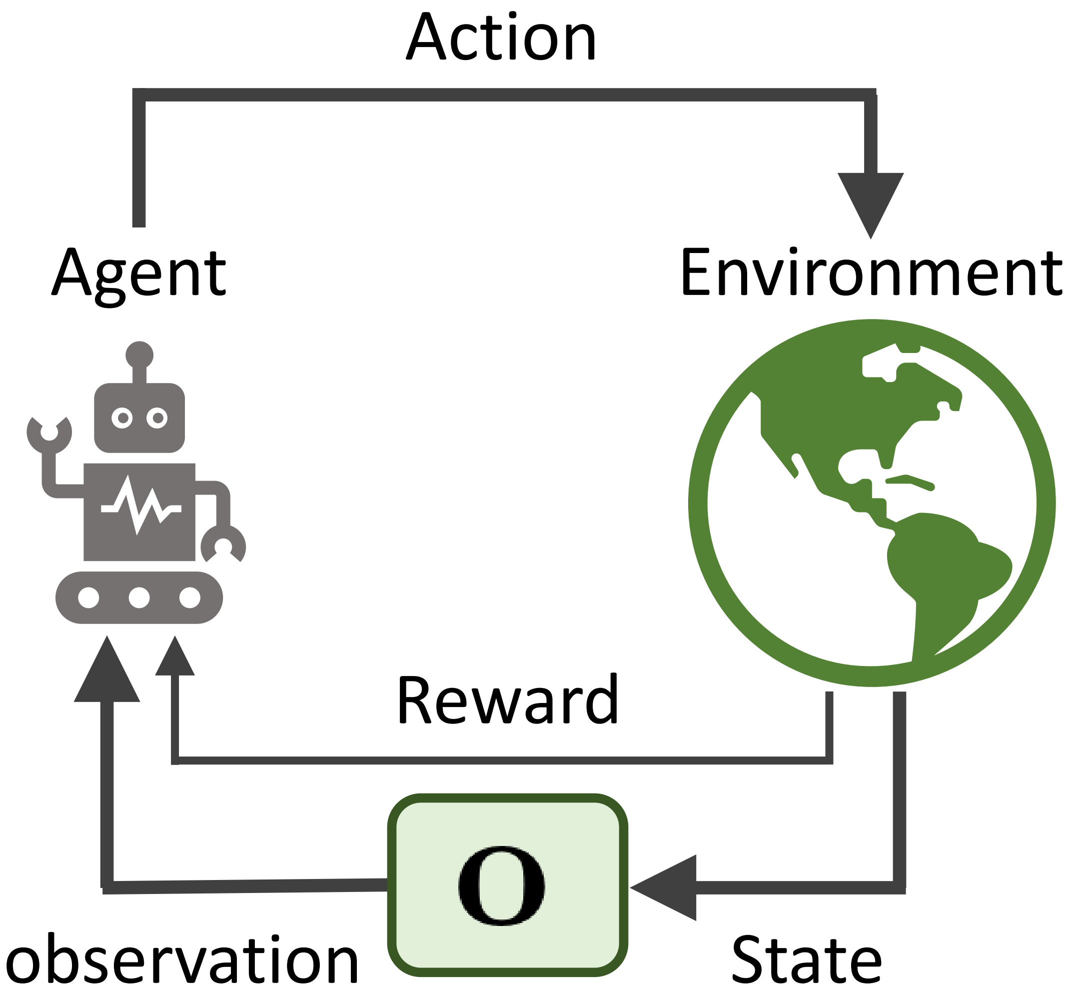
Observations



Agent-Environment Interface



Agent-Environment Interface



Partially Observable Markov Decision Process

$\mathbf{s} \in \mathcal{S}$ – state space

$\mathbf{a} \in \mathcal{A}$ – action space

$p(\mathbf{s}'|\mathbf{s}, \mathbf{a})$ – dynamics function

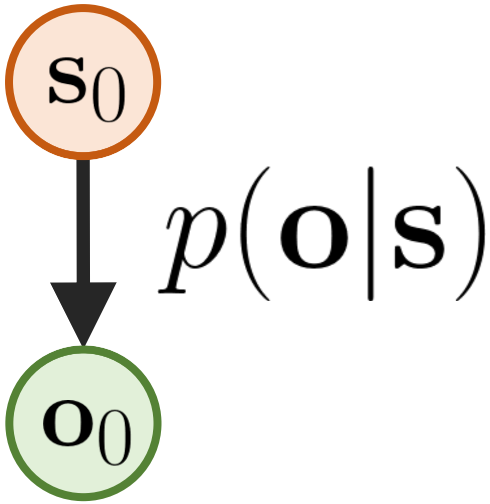
$r(\mathbf{s}, \mathbf{a}, \mathbf{s}')$ – reward function

$\mathbf{o} \in \mathcal{O}$ – observation space

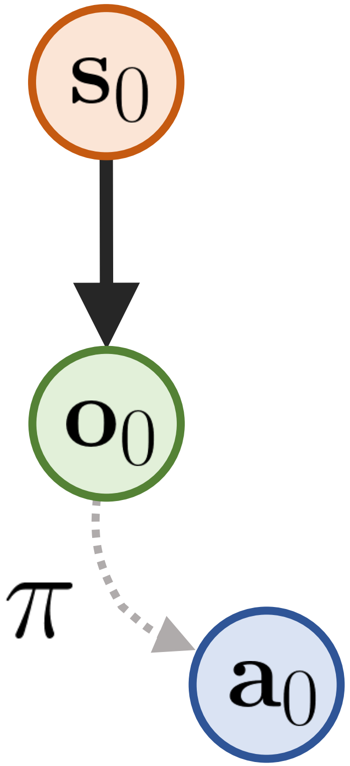
$p(\mathbf{o}|\mathbf{s})$ – observation function

POMDP

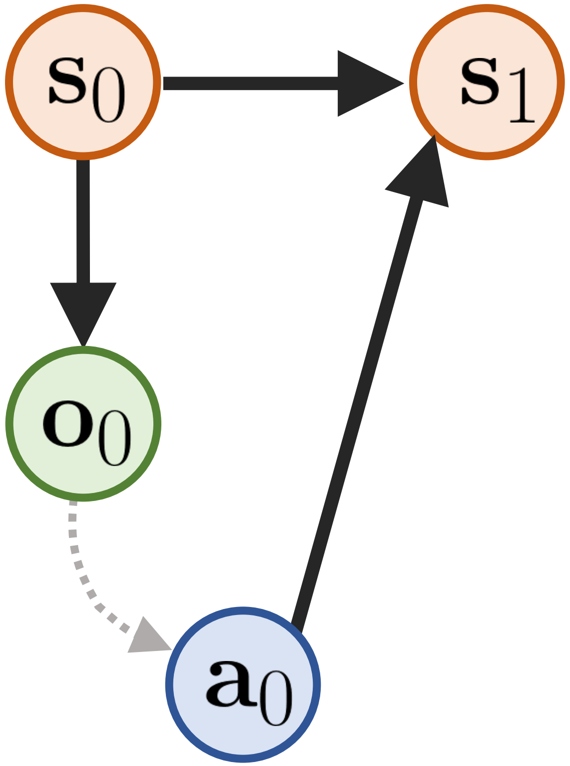
POMDP



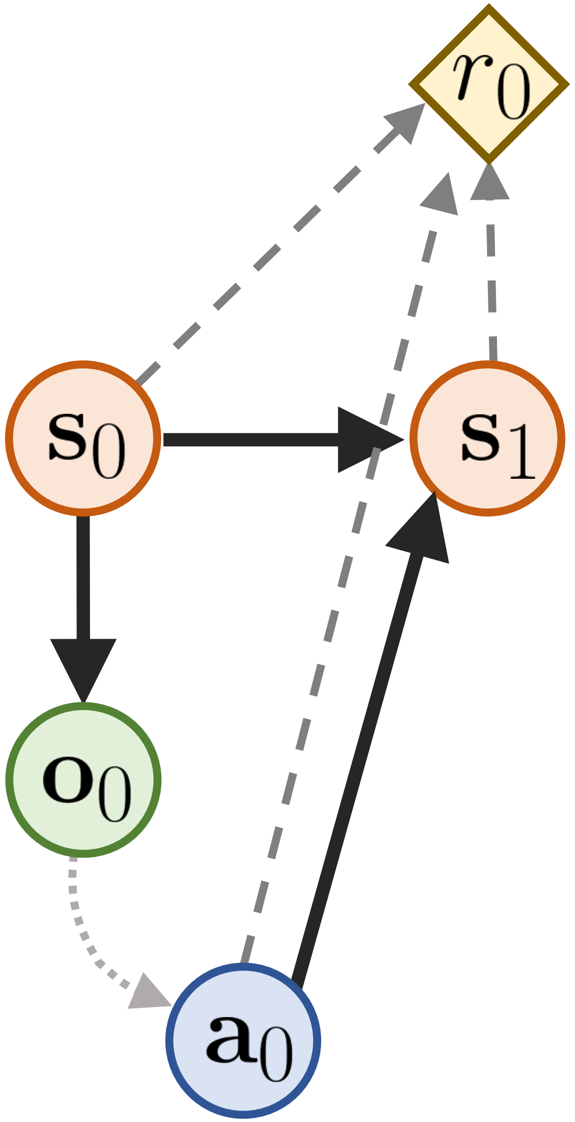
POMDP



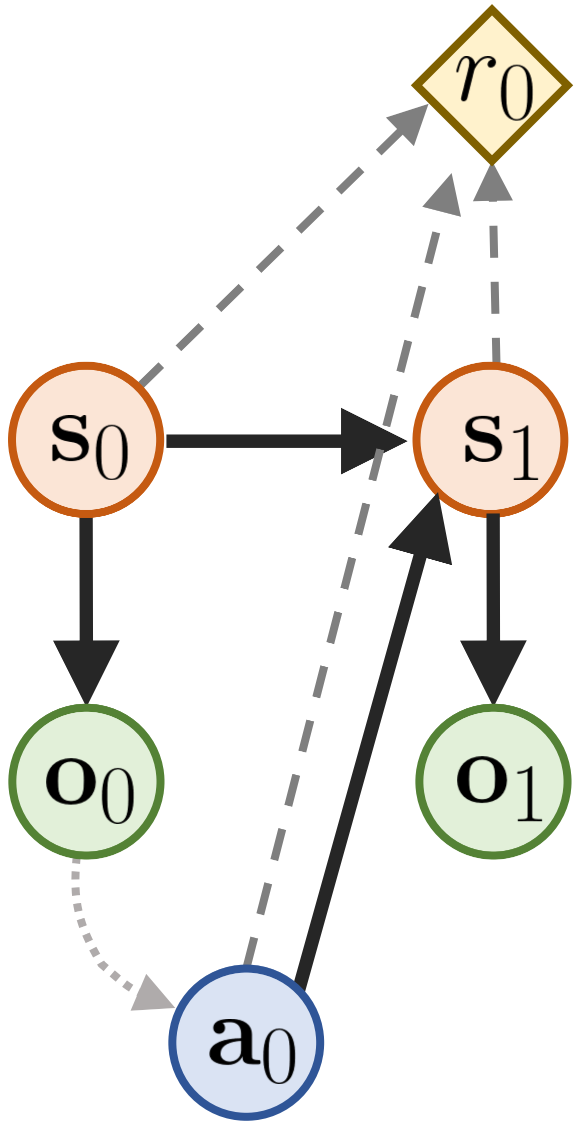
POMDP



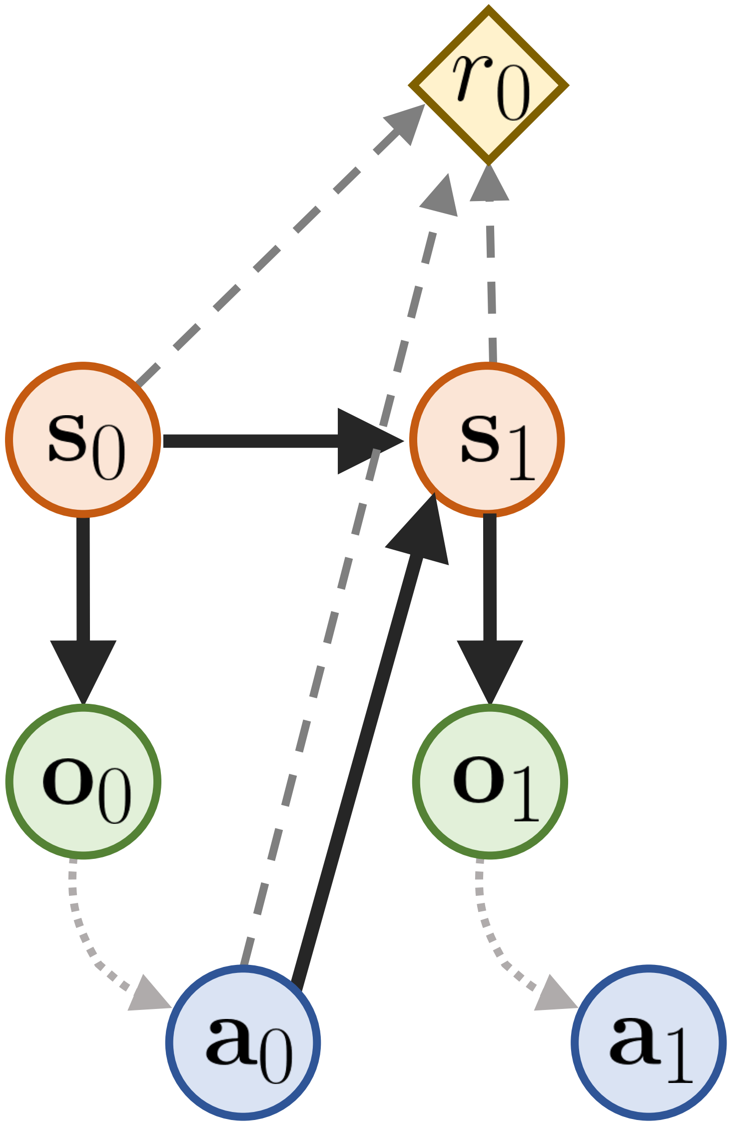
POMDP



POMDP

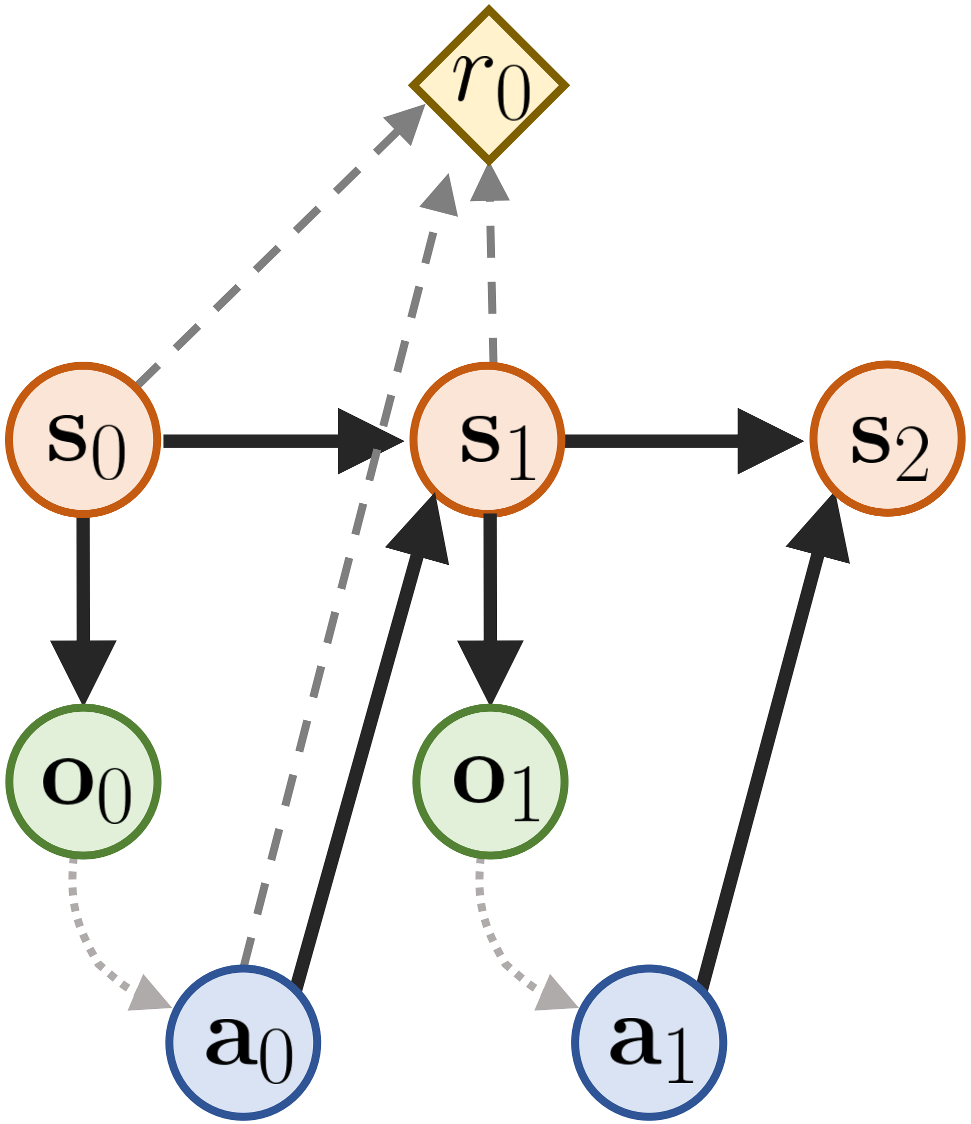


POMDP



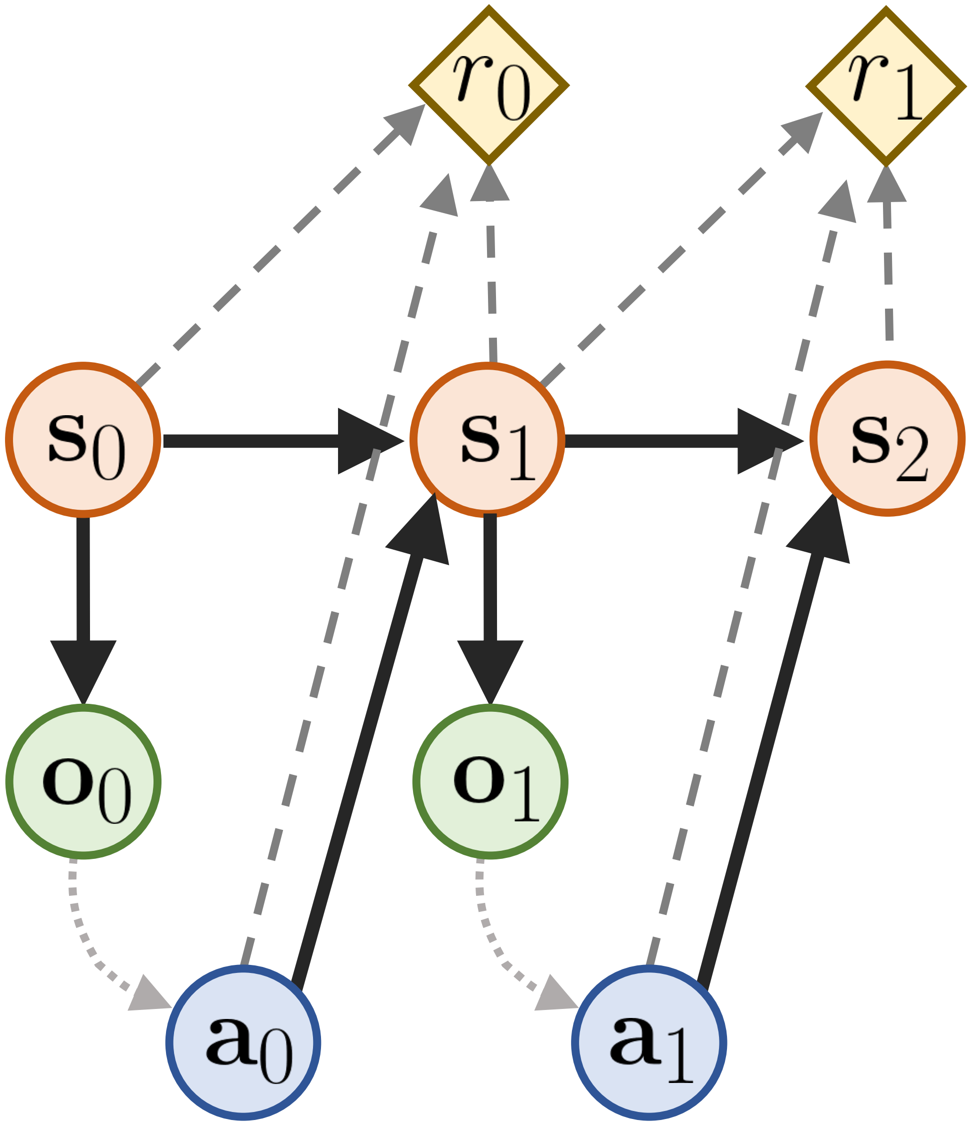
r_0
$\mathbf{s}_0$
$\mathbf{s}_1$
$\mathbf{o}_0$
$\mathbf{o}_1$
$\mathbf{a}_0$
$\mathbf{a}_1$

POMDP



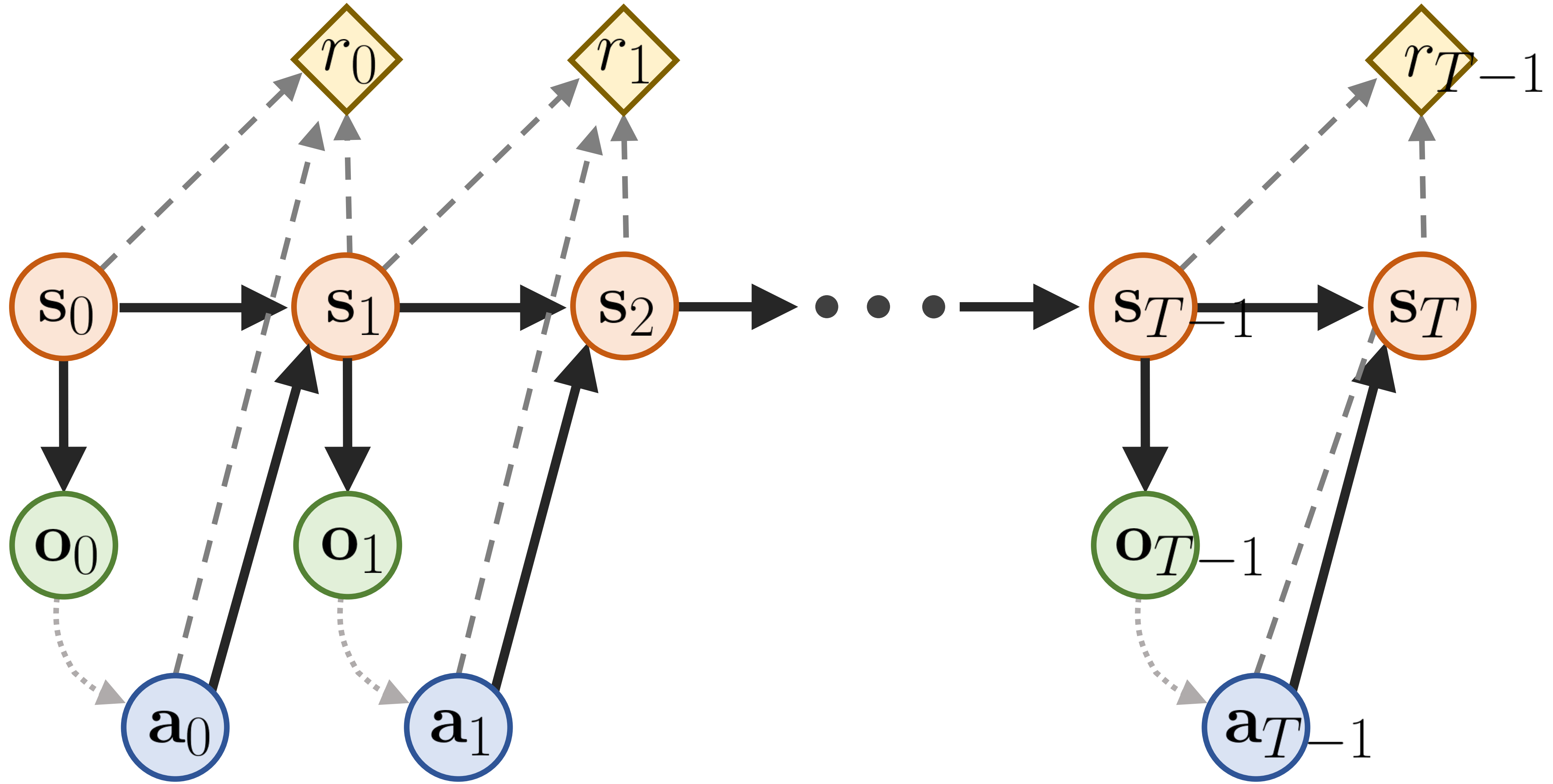
r_0
$\mathbf{s}_0$
$\mathbf{s}_1$
$\mathbf{s}_2$
$\mathbf{o}_0$
$\mathbf{o}_1$
$\mathbf{a}_0$
$\mathbf{a}_1$

POMDP



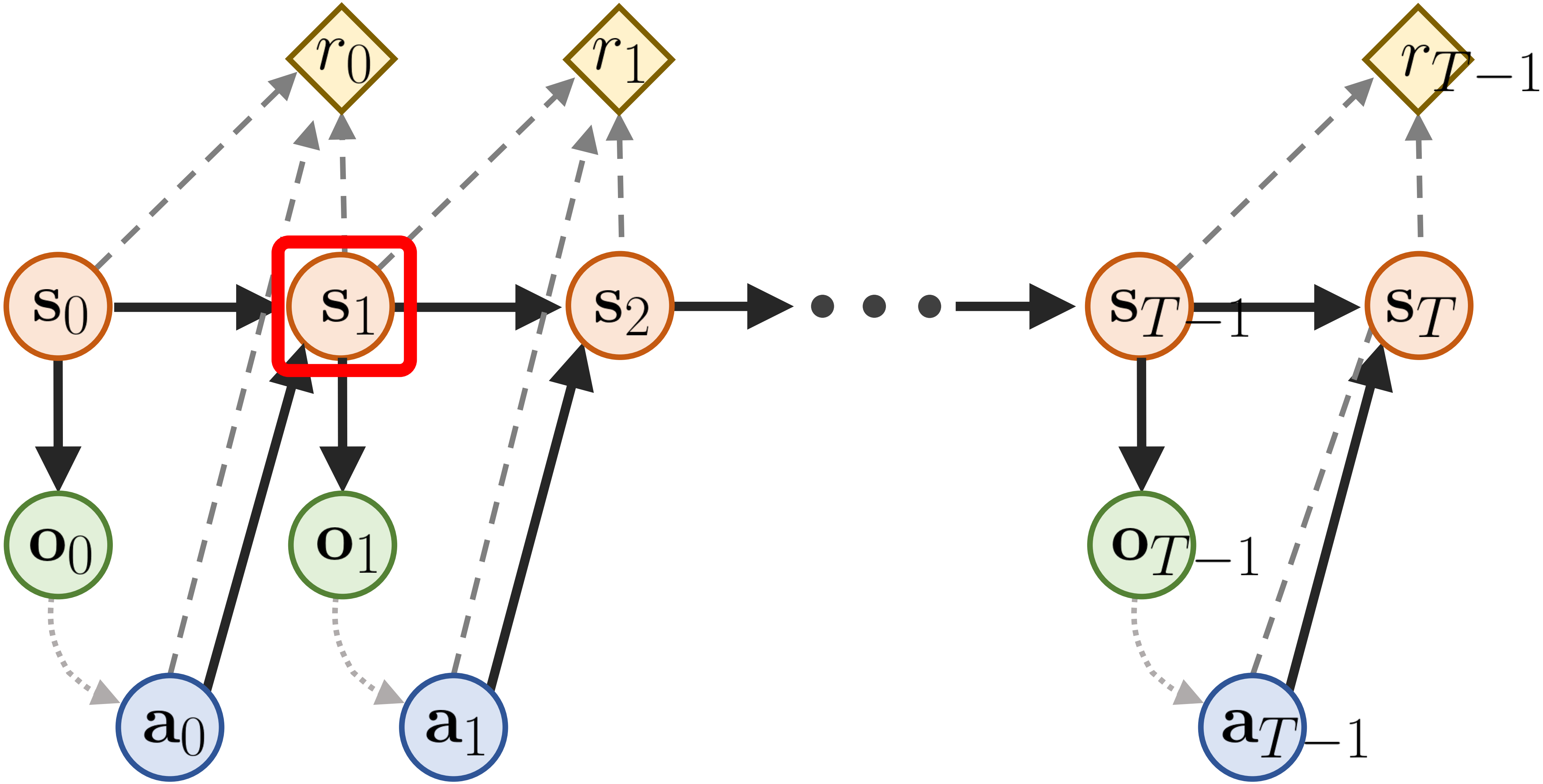
r_0
r_1
$\mathbf{s}_0$
$\mathbf{s}_1$
$\mathbf{s}_2$
$\mathbf{o}_0$
$\mathbf{o}_1$
$\mathbf{a}_0$
$\mathbf{a}_1$

POMDP



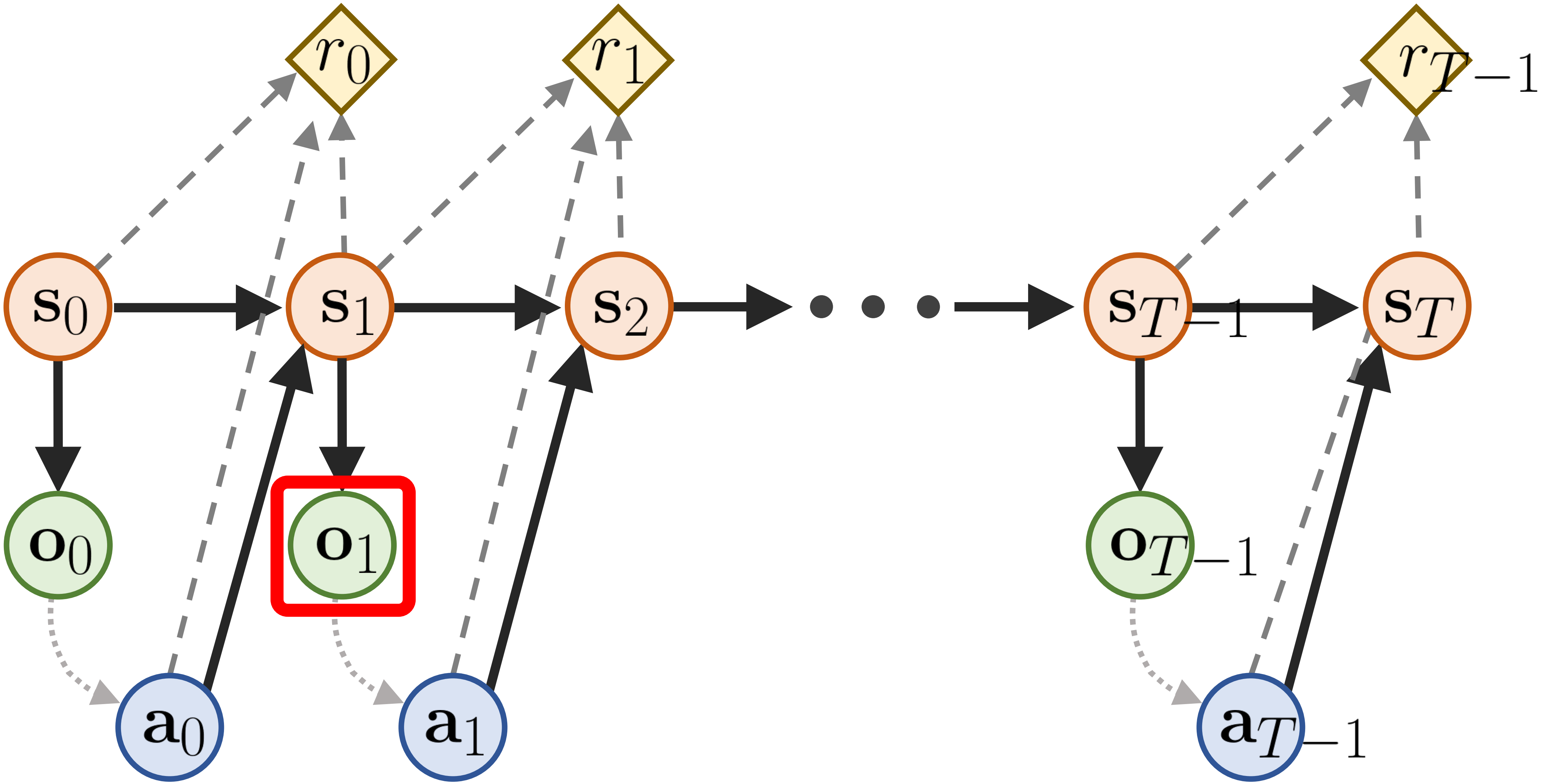
r_0
r_1
r_{T-1}
$\mathbf{s}_0$
$\mathbf{s}_1$
$\mathbf{s}_2$
$\mathbf{s}_{T-1}$
$\mathbf{s}_T$
$\mathbf{o}_0$
$\mathbf{o}_1$
$\mathbf{o}_{T-1}$
$\mathbf{a}_0$
$\mathbf{a}_1$
$\mathbf{a}_{T-1}$

POMDP



r_0
r_1
r_{T-1}
$\mathbf{s}_0$
$\mathbf{s}_1$
$\mathbf{s}_2$
$\mathbf{s}_{T-1}$
$\mathbf{s}_T$
$\mathbf{o}_0$
$\mathbf{o}_1$
$\mathbf{o}_{T-1}$
$\mathbf{a}_0$
$\mathbf{a}_1$
$\mathbf{a}_{T-1}$

POMDP


r_0
r_1
r_{T-1}
$\mathbf{s}_0$
$\mathbf{s}_1$
$\mathbf{s}_2$
$\mathbf{s}_{T-1}$
$\mathbf{s}_T$
$\mathbf{o}_0$
$\mathbf{o}_1$
$\mathbf{o}_{T-1}$
$\mathbf{a}_0$
$\mathbf{a}_1$
$\mathbf{a}_{T-1}$

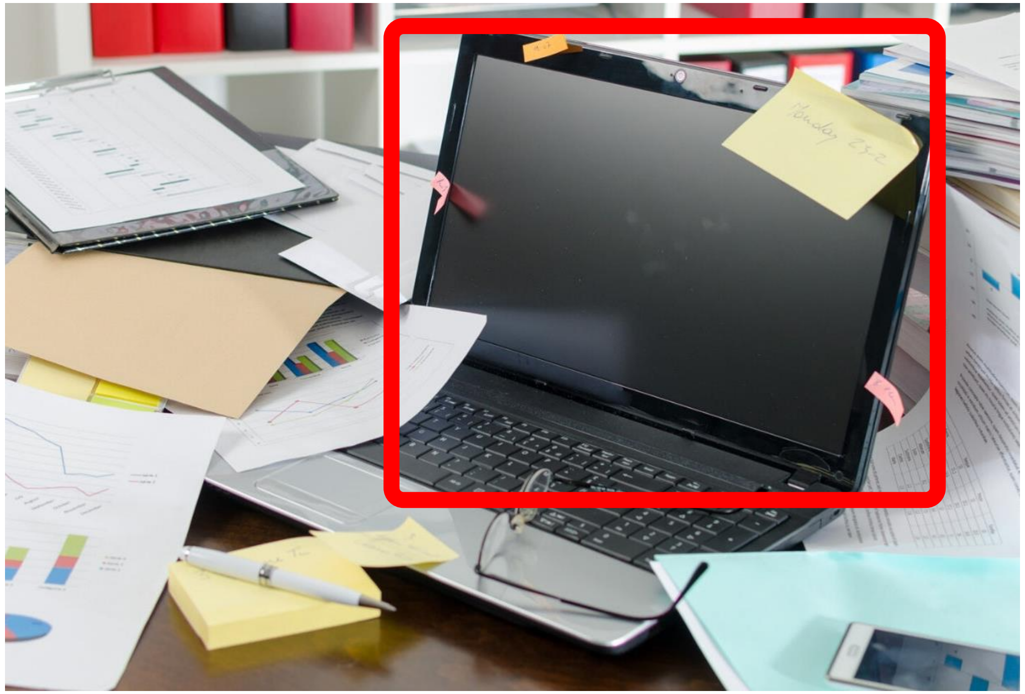
Observation Function

State:

- Location of every item

Observation

- image of the desk



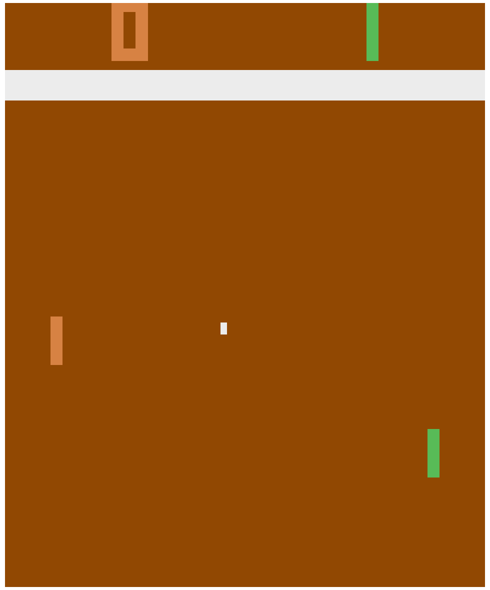
Pong

State:

- position + velocity of paddles
- position + velocity of the ball
- scores

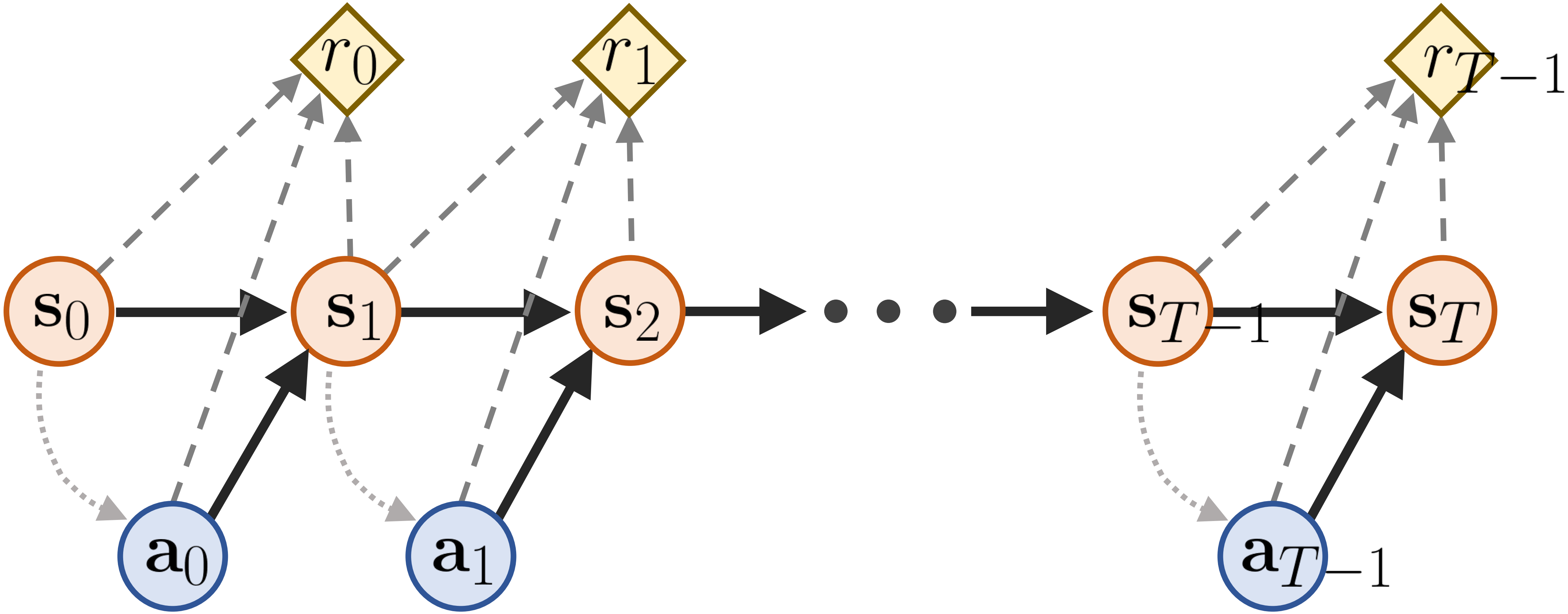
Observation

- image of the game screen



Pong [Atari]

MDP



r_0
r_1
r_{T-1}
$\mathbf{s}_0$
$\mathbf{s}_1$
$\mathbf{s}_2$
$\mathbf{s}_{T-1}$
$\mathbf{s}_T$
$\mathbf{a}_0$
$\mathbf{a}_1$
$\mathbf{a}_{T-1}$

Summary

- Agent-Environment Interface
- Markov Decision Processes
- Partially Observable Markov Decision Processes